Representation formula for stochastic Schrödinger evolution equations and applications

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REPRESENTATION FORMULA FOR STOCHASTIC SCHRÖDINGER EVOLUTION EQUATIONS AND APPLICATIONS

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Abstract. We prove a representation formula for solutions of Schrödinger equations with potentials multiplied by a temporal real-valued white noise in the Stratonovich sense. Using this formula, we obtain a dispersive estimate which allows us to study the Cauchy problem in L^2 or in the energy space of model equations arising in Bose Einstein condensation [1] or in fiber optics [2]. Our results also give a justification of diffusion-approximation for stochastic nonlinear Schrödinger equations.

1. Introduction

The following nonlinear Schrödinger equations perturbed by a potential, deterministic in space and white noise in time have been used as model equations in several applications in Physics.

$$i\partial_t \psi = \frac{1}{2}(-\Delta\psi + V(x)\psi) - i\gamma\psi + \lambda|\psi|^2\psi + \frac{1}{2}K(x)\psi\dot{\xi}(t), \quad t \geq 0, \quad x \in \mathbb{R}^d. \tag{1.1}$$

For example, in [1] and [16], the authors propose the above equation with $V(x) = K(x) = |x|^2$ to describe Bose condensate wave function in all-optical far-off-resonance laser trap, arguing that fluctuations of the laser intensity are observed in this case. In this model, the term $\dot{\xi}(t)$ represents the relative deviations of the laser intensity $E(t)$ around its mean value (see [1]), and is assumed to be a real-valued white noise in time with correlation function $\mathbb{E}(\dot{\xi}(t)\dot{\xi}(s)) = \sigma_0^2\delta_0(t-s)$. Here, δ_0 denotes the Dirac measure at the origin, and $\sigma_0 \in \mathbb{R}$. The damping term, with a coefficient $\gamma \geq 0$, describes the interactions with the thermal cloud created by non-condensed atoms. Finally, the sign of λ is related to the sign of the atomic scattering length, which may be positive or negative, and it may be assumed without loss of generality that $\lambda = \pm 1$.

Related equations may also be found in the context of optic fibers. In [2] e.g., equation (1.1) without the potential in the drift but with a multiplicative noise, i.e., $V(x) = 0$, $K(x) = |x|^2$, and $\dot{\xi}(t)$ as above, was considered as a model for optical soliton propagation in fibers with random inhomogeneities.

Our aim in this paper is, in order to justify these model equations (1.1) from the mathematical point of view, first to construct the fundamental solution of (1.1) with $\lambda = \gamma = 0$, and establish the corresponding dispersive estimates. This result will then enable us to prove the global existence of solutions of Eq. (1.1) (with a more general nonlinear term) in L^2, in subcritical cases, since the L^2 norm of the evolution

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equation is bounded if $\gamma \geq 0$. For this purpose, we need some "good" properties of the integral kernel of the linear evolution propagator (with $\gamma = 0$), which can be expressed in terms of classical orbits, as is often used in semiclassical analysis. Using these classical paths, we can write the propagator as an oscillatory integral operator associated to the action integral. Such oscillatory integral operators have been studied by many authors in the context of deterministic Schrödinger equations (see e.g. [13, 14, 15, 18, 20, 28]). In the present paper we follow Yajima [28] who derived dispersive estimates for Schrödinger equations with magnetic fields. We use the following gauge transformation. We define $G(t,x) = \frac{1}{2}(V(x)t + K(x)\xi(t))$ and consider the change of gauge:

$$\psi(t,x) = e^{-iG(t,x)}u(t,x) \tag{1.2}$$

where $\psi(t,x)$ verifies Eq. (1.1) with $\gamma = \lambda = 0$. After this transformation, u satisfies the following Schrödinger equation with a random magnetic field:

$$i\partial_t u = -\frac{1}{2}\sum_{j=1}^{d}(\partial_{x_j} - iA_j(t,x))^2 u, \quad A = \nabla G(t,x) = \frac{1}{2}(\nabla V(x)t + \nabla K(x)\xi(t)). \tag{1.3}$$

The theory of [28] does not apply directly to equation (1.3), since it requires that the time derivative of the vector potential $A(t,x)$ is uniformly bounded, while this time derivative only exists as a distribution in our case, since $\dot{\xi}(t)$ is a white noise. We will however prove, making use of the almost sure C^α regularity of the Brownian motion, with $0 < \alpha < 1/2$, that the estimates in [28] can be generalized to our case. Actually, in our study, $\xi(t)$ could be replaced by any real valued C^α function of the time variable, with $\alpha > 0$. After having completed this work, we were told about the existence of the paper [23] where an explicit formula is given for solutions of linear equations of the form (1.3) with purely quadratic Hamiltonian and continuous coefficients in time. However, with our extra regularity assumptions in time (C^α instead of C^0) we get a slightly more precise description for small times (see (3.21) below). We will see below that this regularity also allows us to prove the continuous dependence of the solution of (1.1) on the Brownian paths.

Some linear stochastic equations similar to (1.1) with $\lambda = 0$ have been studied in the context of stochastic quantum mechanics. In [26], e.g. an equation of the form (1.1) (with $\lambda = 0$), but with in addition a stochastic magnetic field is considered in the semi-classical limit, and semi-classical expansions at any order are given. In [29], a representation formula using Fresnel type path integral is given for the solution of (1.1) with $\lambda = 0$, when $V(x)$ and $K(x)$ are Fourier transforms of bounded complex Borel measures on $\mathbb{R}^d$ (this is clearly not the case in our situation). This representation is similar to that given in [3] for deterministic linear Schrödinger equations. However, it is not clear whether this representation, which involves an integral on infinite dimensional space, would lead to Strichartz estimates as those we use here to study the nonlinear equation (see Proposition 7 below).

We are also interested in the convergence, as ε tends to zero, of the solution of the following equation to Eq. (1.1),

$$i\partial_t \varphi = \frac{1}{2}(-\Delta + V(x))\varphi - i\gamma\varphi + \lambda|\varphi|^2\varphi + \frac{1}{2\varepsilon}m\Big(\frac{t}{\varepsilon^2}\Big)K(x)\varphi, \quad t \geq 0, \quad x \in \mathbb{R}^d \tag{1.4}$$

where $m(t)$ is a centered stationary random process, and $\sigma_0^2 = 2\mathbb{E}\int_0^{+\infty} m(0)m(t)dt$. Garnier, Abdullaev and Baizakov in [16] studied this type of diffusion approximation limit in order to investigate the collapse time of the Bose-Einstein Condensate. They use this analysis for the differential equations of the action-angle variables in order to explicit the structure of the width of the BEC, which satisfy a closed form ODE in the variational ansatz. The same kind of study has been performed in [11, 21] for some model equations in optical fibers with dispersion management. We will address this diffusion-approximation for Eq. (1.1), but only in the subcritical cases (see Remark 2.4).

In order to state precisely the problem and our results, we consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with a standard filtration $(\mathcal{F}_t)_{t\geq 0}$ such that $\mathcal{F}_0$ is complete, and a standard real valued Brownian motion $W(t)$ on $\mathbb{R}^+$ starting at 0, associated with the filtration $(\mathcal{F}_t)_{t\geq 0}$. We set $\dot{\xi} = \sigma_0 \frac{dW}{dt}$ and then consider the stochastic nonlinear Schrödinger equation with a more general nonlinear term than Eq. (1.1)

$$id\psi + \frac{1}{2}(\Delta\psi - V(x)\psi)dt - \lambda|\psi|^{2\sigma}\psi dt = \frac{\sigma_0}{2}K(x)\psi \circ dW, \tag{1.5}$$

where $\sigma > 0$, $\sigma_0 \in \mathbb{R}$, $\lambda = \pm 1$, and $\circ$ stands for the Stratonovich product in the right hand side of (1.5), which is natural since the noise here arises as the limit of processes with nonzero correlation length. Note that we set $\gamma = 0$ for the sake of simplicity, but the global existence and convergence results of Theorems 2 and 3 can be easily generalized to the case $\gamma > 0$ (see Remarks 4.1 and 5.1).

We define

$$\Sigma(k) = \Big\{ v \in L^2(\mathbb{R}^d), \sum_{|\alpha|+|\beta|\leq k} |x^\beta \partial_x^\alpha v|_{L^2}^2 = |v|_{\Sigma(k)}^2 < +\infty \Big\}$$

for $k \in \mathbb{N}$, and write $\Sigma(-k)$ for the dual space of $\Sigma(k)$ in the L^2 sense. In particular we denote $\Sigma(1)$ by Σ.

Note that for the deterministic case $V(x) = |x|^2$ and $K \equiv 0$, it is known that Eq.(1.5) is locally well posed in Σ, for $\lambda = \pm 1$, $\sigma < \frac{2d}{d-2}$ if $d \geq 3$ or $\sigma < +\infty$ if $d = 1, 2$ and globally well posed if either $\lambda = 1$ or $\lambda = -1$ and $\sigma < 2/d$ (see Oh [24]). These results in the deterministic case may be proved with the help of the dispersive estimate for small time : for $p \in [2, \infty]$,

$$|U_0(t)f|_{L^p(\mathbb{R}^d)} \leq C|t|^{-d(1/2-1/p)}|f|_{L^{p'}(\mathbb{R}^d)}, \quad f \in L^{p'}(\mathbb{R}^d), \tag{1.6}$$

where $U_0(t)$ is the propagator of Eq. (1.5) with $V(x) = |x|^2$, $K \equiv 0$ and $\lambda = 0$. This estimate is obtained, for example, by using the transformation which connects Eq.(1.5) with $V(x) = |x|^2$ to Eq.(1.5) with $V(x) = 0$ for the case of $K = \lambda \equiv 0$:

$$u(t, x) = \frac{1}{(\cos t)^{d/2}} e^{-\frac{i}{2}x^2 \tan t} v\Big(\tan t, \frac{x}{\cos t}\Big),$$

where v is the solution of Eq. (1.5) with $V = K = \lambda \equiv 0$ (see e.g. [5]). However, this transformation does not seem useful in the stochastic case, i.e., in the case where $K \not\equiv 0$. Using a compactness method, we generalized in [12] the deterministic existence and uniqueness results to Eq. (1.5), but only in space dimensions one and two (and with restrictions on σ) due to the lack of dispersive estimate of the form (1.6). In the present paper, we prove such a dispersive estimate for equation (1.5) with $\lambda = 0$. As a consequence, we will improve the results in [12], showing some existence results in $d \geq 3$. We then prove the continuity of the solution on the Brownian path in the subcritical case $\sigma < 2/d$ and deduce the convergence of the solution of (1.4) to the solution of (1.1) as ε goes to zero.

Let us give some notations. We denote by e_k $(1 \leq k \leq d)$ the unit vector pointing in the direction of the x_k axis in $\mathbb{R}^d$. The number p' is the conjugate of $p \in [1, \infty]$ given by $\frac{1}{p} + \frac{1}{p'} = 1$. In all the paper, $\Theta_M \in C_0^\infty(\mathbb{R}^d)$ is a cut-off function with $\mathrm{supp}\Theta_M \subset \{x \in \mathbb{R}^d, |x| \leq 2M\}$ and $\Theta_M \equiv 1$ on $\{x \in \mathbb{R}^d, |x| \leq M\}$ for $M > 0$.

If I is an interval of $\mathbb{R}$, E is a Banach space, and $1 \leq r \leq \infty$, then $L^r(I, E)$ is the space of strongly Lebesgue measurable functions v from I into E such that the function $t \to |v(t)|_E$ is in $L^r(I)$. We define similarly the space $C(I, E)$. The inner product in the Hilbert space $L^2(\mathbb{R}^d)$ is denoted by $\langle \cdot, \cdot \rangle$, i.e., $\langle u, v \rangle = \int_{\mathbb{R}^d} u(x)\bar{v}(x)dx$ for $u, v \in L^2(\mathbb{R}^d)$.

The paper is organized as follows. In Section 2, we mention precisely our results. In Section 3, we study the linear problem. We first give some properties of the classical orbits associated with the Schrödinger operator with magnetic field $\frac{1}{2}(\nabla - iA(t,x))^2$. Using these properties, we define the action functional, and we construct the integral kernel of the oscillatory integral propagator. Note that we give only outlines in Section 3, the reader will find the proofs for this section in the appendix. In Section 4 we prove the existence of solutions of a modified equation (see Proposition 1 below), which will immediately give the proof of the existence of solutions for (1.5). Section 5 is devoted to the continuous dependence of solutions on the Brownian paths in $L^2(\mathbb{R}^d)$. Using this latter result, we also show the convergence of the solution of (1.4) to the solution of (1.5) in distribution in $C([0,T];L^2(\mathbb{R}^d))$, as ε goes to zero. To lighten notations, we denote sometimes in what follows by $C_{\theta,\cdots}$ a constant which depends on θ and so on.

2. Main results

First, we mention our results for linear Schrödinger equations. We will deal with the specific case where $V(x) = \sum_{j=1}^d \nu_j x_j^2$, and $K(x) = \sum_{j=1}^d \kappa_j x_j^2$, with ν_j and $\kappa_j \in \mathbb{R}$. Then, using transformation (1.2) with

$$G(t,x) = \frac{1}{2}(V(x)t + \sigma_0 K(x)W(t)), \tag{2.1}$$

we are led to consider the equation (1.3), with, for $1 \le j \le d$,

$$A_j(t,x) = \frac{1}{2}(\partial_{x_j}V(x)t + \sigma_0\partial_{x_j}K(x)W(t)) = x_j(\nu_j t + \sigma_0\kappa_j W(t)).$$

Remark 2.1. *For each $t \ge 0$, and each ω such that $W(\cdot,\omega)$ is continuous at t, the linear operator $H_\omega(t) = \Big(\nabla - iA(t,x)\Big)^2$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^d)$, and its closure is identical with its maximal extension which is denoted by the same symbol (see, e.g.* [25]*, Theorem X.34). The domain of $H_\omega(t)$ is given, for each $t \ge 0$, by*

$$D(H_\omega(t)) = \{v \in L^2(\mathbb{R}^d), H_\omega(t)v \in L^2(\mathbb{R}^d)\},$$

and contains the space $\Sigma(2)$.

We now state our result on the propagator of the linear evolution equation (1.3).

Theorem 1. *Let $T_0 > 0$ and $0 < \alpha < 1/2$ be fixed, and let $\omega \in \Omega$ be such that $W(\cdot,\omega) \in C^\alpha([0,T_0])$. There exists a positive number T_ω and a unique propagator $\{U^\omega(t,s), t,s \in [0,T_0], |t-s| \le T_\omega\}$ with the following properties.*

(i) *$U^\omega(t,s)$ can be written in the form of an oscillatory integral operator as follows :*

$$U^\omega(t,s)f(x) = (2\pi i(t-s))^{-d/2}a(t,s)\int_{\mathbb{R}^d} e^{iS(t,s,x,y)}f(y)dy, \quad f \in C_0^\infty(\mathbb{R}^d),$$

where $a(t,s)$ is a C^1 function, depending on ω, of $t,s \in [0,T_0]$ with $|t-s| \le T_\omega$ satisfying $|a(t,s)-1| \le C_{\omega,T_0}|t-s|$ for some constant C_{ω,T_0}. The real valued phase function $S(t,s,x,y)$ depending on ω satisfies the Hamilton-Jacobi equations:

$$(\partial_t S)(t,s,x,y) + (1/2)\big((\nabla_x S)(t,s,x,y) - A(t,x)\big)^2 = 0,$$
$$(\partial_s S)(t,s,x,y) - (1/2)\big((\nabla_y S)(t,s,x,y) + A(s,y)\big)^2 = 0,$$

and the following property : for any multi-index γ,β, $\partial_x^\gamma\partial_y^\beta S \equiv 0$ if $|\gamma+\beta| \ge 3$ and

$$\Big|\partial_x^\gamma\partial_y^\beta\Big\{S(t,s,x,y) - \frac{|x-y|^2}{2(t-s)}\Big\}\Big| \le C_{\gamma,\beta,\omega,T_0}, \quad \text{if} \quad |\gamma+\beta| = 2.$$

(ii) *The operator* $U^\omega(t,s)$ *is a linear, unitary operator in* $L^2(\mathbb{R}^d)$*, and satisfies*

$$U^\omega(t,s) = U^\omega(t,h)U^\omega(h,s), \quad \textit{for} \quad 0 \le s < h < t \le T_0, \quad |t-s| \le T_\omega.$$

Moreover, if $f \in L^2(\mathbb{R}^d)$*, then* $U^\omega(\cdot,s)f$ *is continuous in* t *with values in* $L^2(\mathbb{R}^d)$*, and* $\partial_t U^\omega(\cdot,s)f$ *is continuous with values in* $\Sigma(-2)$ *and satisfies*

$$i\partial_t U^\omega(t,s)f = -\frac{1}{2}\big(\nabla - iA(t,x)\big)^2 U^\omega(t,s)f, \quad \textit{in} \quad \Sigma(-2).$$

Remark 2.2. *We could construct the propagator of (1.3) for more general potentials* $V(x)$ *and* $K(x)$*, for example for smooth real-valued* $V(x)$*,* $K(x)$ *satisfying*

$$\sup_{x\in\mathbb{R}^d} |\partial_x^\alpha V(x)|, \sup_{x\in\mathbb{R}^d} |\partial_x^\alpha K(x)| \le C_\alpha$$

for any multi-index α *with* $|\alpha| \ge 2$*. For the construction, one could follow the same arguments as in* [28]*, which uses the approximation of the propagator* $U^\omega(t,s)$ *by the semi-classical propagator whose amplitude function is defined as the series of solutions to the associated transport equation. The iteration procedure would be justified similarly to* [13, 14, 28]*, making use of Kumanogo-Taniguchi techniques in* [19] *for multi-product of pseudo-differential operators. In the case where* $V(x) = \sum_{j=1}^d \nu_j x_j^2$*, and* $K(x) = \sum_{j=1}^d \kappa_j x_j^2$*, the system satisfied by the classical paths is linear (see (3.1) below), thus only the first term in the series is nonvanishing.*

In order to apply the above results to the nonlinear equation (1.5), we first solve the following equation for u, which is related to ψ by (1.2) :

$$i\partial_t u = -\frac{1}{2}\Big(\nabla - iA(t,x)\Big)^2 u + \lambda |u|^{2\sigma} u. \tag{2.2}$$

More precisely, we consider the mild form of Eq. (2.2) which, as is well known, is equivalent to equation (2.2) as long as we consider solutions which are, at least, continuous in time with values in $L^2(\mathbb{R}^d)$; the initial data $u(0) = u_0$ is in $L^2(\mathbb{R}^d)$.

$$u(t) = U^\omega(t,0)u_0 - i\lambda \int_0^t U^\omega(t,s)|u(s)|^{2\sigma} u(s)ds. \tag{2.3}$$

Remark 2.3. *Here, we have defined* $U^\omega(t,s)$ *for any* $t,s \in [0,T_0]$ *by setting* $U^\omega(t,s) = U^\omega(s_n, s_{n-1}) \circ \cdots \circ U^\omega(s_1,s_0)$ *where* $[s,t]$ *has been decomposed into* $[s_0,s_1] \cup [s_1,s_2] \cup \cdots \cup [s_{n-1},s_n]$ *with* $s = s_0$ *and* $t = s_n$ *so that* $|s_{j+1} - s_j| \le T_\omega$*,* $0 \le j \le n-1$*.*

Equations of the form (2.2), but with magnetic vector potentials A independent of time or with bounded time derivatives have been studied e.g. in [9, 22] using the results of [28] on the propagator of the linear equation. Here, we generalize those results to our case, by using standard arguments in the deterministic theory (see e.g. [17, 27]).

Proposition 1. *Assume* $\sigma > 0$ *and* $\lambda = \pm 1$*. Let* $2/r = d(1/2 - 1/(2\sigma+2))$*.*

(i) *Let* $u_0 \in L^2(\mathbb{R}^d)$ *and* $\sigma < 2/d$*. Then there exists a unique global solution* u *of (2.3), adapted to* $(\mathcal{F}_t)_{t\ge0}$*, almost surely in* $C([0,T_0]; L^2(\mathbb{R}^d)) \cap L^r(0,T_0; L^{2\sigma+2}(\mathbb{R}^d))$ *for any* $T_0 > 0$*. Moreover, the* L^2 *norm is conserved:*

$$|u(t)|_{L^2} = |u(0)|_{L^2}, \quad \textit{a.s. in } \omega, \quad \textit{for all } t \ge 0,$$

and u *depends continuously on the initial data* u_0 *in the following sense: if* $u_{0,n} \to u_0$ *in* $L^2(\mathbb{R}^d)$*, and if* u_n *denotes the solution of (2.3) with* u_0 *replaced by* $u_{0,n}$*, then* $u_n \to u$ *in* $L^\infty(0,T_0;L^2)$*.*

(ii) *Let $u_0 \in \Sigma$ and $\sigma < 2/d$. Then there exists a unique global adapted solution u of (2.3) almost surely in $C(\mathbb{R}^+;\Sigma)$.*

(iii) *Let $u_0 \in \Sigma$, $\sigma < 2/(d-2)$ if $d \geq 3$ and $\sigma < +\infty$ if $d = 1, 2$. Then there exists a maximal time $T^* = T^*_{u_0,\omega} > 0$ such that there exists a unique adapted solution $u(t)$ of (2.3) almost surely in $C([0,T^*);\Sigma)$, and the following alternative holds: $T^* = +\infty$ or $T^* < +\infty$ and $\lim_{t\uparrow T^*} |u(t)|_\Sigma = +\infty$.*

Transformation (1.2) changes only the phase of the solution, so it preserves the form of the nonlinear term. Also since the solution $u(t)$ of (2.2) given by Proposition 1 is adapted, so is $\psi(t)$ (see (1.2)). We then obtain the following results concerning equation (1.5).

Theorem 2. *Assume $\sigma > 0$ and $\lambda = \pm 1$. Let $2/r = d(1/2 - 1/(2\sigma+2))$.*

(i) *Let $\psi_0 \in L^2(\mathbb{R}^d)$ and $\sigma < 2/d$. Then there exists a unique global solution $\psi(t)$ of (1.5), adapted to $(\mathcal{F}_t)_{t\geq 0}$ with $\psi(0) = \psi_0$, which is almost surely $C(\mathbb{R}^+;L^2(\mathbb{R}^d)) \cap L^r_{loc}(\mathbb{R}^+;L^{2\sigma+2})$. Moreover, the L^2 norm is conserved by the time evolution, that is,*

$$|\psi_0|_{L^2} = |\psi(t)|_{L^2}, \quad \text{a.s. in } \omega, \quad \text{for all } t \geq 0.$$

(ii) *Let $\psi_0 \in \Sigma$ and $\sigma < 2/d$. Then there exists a unique global solution $\psi(t)$ of (1.5), adapted to $(\mathcal{F}_t)_{t\geq 0}$ with $\psi(0) = \psi_0$, almost surely in $C(\mathbb{R}^+;\Sigma)$.*

(iii) *Let $\psi_0 \in \Sigma$ and $\sigma < 2/(d-2)$ if $d \geq 3$, $\sigma < +\infty$ if $d = 1, 2$. Then there exist a stopping time $\tau^* = \tau^*_{\psi_0,\omega} > 0$ and a unique solution $\psi(t)$ of (1.5), adapted to $(\mathcal{F}_t)_{t\geq 0}$ with $\psi(0) = \psi_0$, almost surely in $C([0,\tau^*);\Sigma)$. In fact, $\tau^* = T^*$, defined in Proposition 1 (iii).*

(iv) *Let $\lambda = 1$, $\psi_0 \in \Sigma$ and $\sigma < 2/(d-2)$ if $d \geq 3$, $\sigma < +\infty$ if $d = 1, 2$. Then there exists a unique global solution $\psi(t)$ of (1.5) adapted to $(\mathcal{F}_t)_{t\geq 0}$, with $\psi(0) = \psi_0$, almost surely in $C(\mathbb{R}^+;\Sigma)$.*

In (iv) above, we use the Hamiltonian

$$H(\psi) = \frac{1}{4}|\nabla\psi|^2_{L^2} + \frac{1}{4}\sum_{j=1}^{d} \nu_j |x_j\psi|^2_{L^2} + \frac{\lambda}{2\sigma+2}|\psi|^{2\sigma+2}_{L^{2\sigma+2}}, \tag{2.4}$$

which is a conserved quantity of the deterministic equation, that is (1.5) with $K \equiv 0$. $H(\psi)$ is well defined for ψ in Σ, thanks to the embedding $\Sigma \subset H^1(\mathbb{R}^d) \subset L^{2\sigma+2}(\mathbb{R}^d)$, for $\sigma < \frac{2d}{d-2}$ if $d \geq 3$ or $\sigma < +\infty$ if $d = 1, 2$.

At last, we prove that equation (1.5) is the diffusion-approximation limit of the equation driven by a stationary process $m(t)$. We will assume the following.

Assumption (A). The real valued centered stationary random process $m(t)$ has trajectories a.s. in $L^\infty(0,T)$ for any $T > 0$, and is such that for any $T > 0$, the process $t \mapsto \frac{\varepsilon}{\sigma_0}\int_0^{t/\varepsilon^2} m(s)ds$ converges in distribution in $C([0,T])$ to a standard real valued Brownian motion as ε tends to zero.

Remark 2.4. *It is classical that such an assumption holds if e.g. m is a homogeneous, centered, stationary and ergodic Markov process such that $\sigma_0^2 = 2\int_0^{+\infty} \mathbb{E}(m(0)m(t))dt < +\infty$. Two classical examples are given by*

(i) *$m(t) = m_n$ for $t \in [n, n+1)$, where (m_n) is an iid family of random variables with finite second moment*

(ii) *m is a Ornstein-Uhlenbeck process, i.e. a stationary solution of*

$$dX = -\lambda X dt + dW$$

with $\lambda > 0$ fixed, and W a real valued Brownian motion.

Theorem 3. *Let $0 < \sigma < 2/d$ and $\lambda = \pm 1$. Suppose that $m(t)$ satisfies Assumption (A) above. Then, for any $\varepsilon > 0$ and $\psi_0 \in L^2(\mathbb{R}^d)$ there exists a unique solution φ_ε, with continuous paths on $\mathbb{R}^+$ with values in $L^2(\mathbb{R}^d)$, of the following equation:*

$$\begin{cases} i\partial_t \varphi = & \frac{1}{2}(-\Delta + V(x))\varphi + \lambda|\varphi|^{2\sigma}\varphi + \frac{1}{2\varepsilon} m\Big(\frac{t}{\varepsilon^2}\Big) K(x)\varphi, \\ \varphi(0) = & \psi_0. \end{cases} \tag{2.5}$$

Moreover for any fixed $T > 0$, the process φ_ε converges in distribution in $C([0,T]; L^2(\mathbb{R}^d))$ as ε tends to zero, to the solution ψ of (1.5).

Theorem 3 is proved as in [11, 21] , by making use of the following proposition concerning the continuous dependence of the solution of (2.2) on the Brownian paths $W(\cdot, \omega)$.

Proposition 2. *Assume $0 < \sigma < d/2$. Let $T_0 > 0$ and $0 < \alpha < 1/2$ be fixed, and for $R > 0$, let B_R be the closed ball of radius R in $C^\alpha([0, T_0])$. Then, for any $u_0 \in L^2(\mathbb{R}^d)$, the mapping*

$$\begin{array}{ccc} W & \mapsto & u^W \\ B_R & \to & C([0,T_0]; L^2(\mathbb{R}^d)) \end{array}$$

is continuous, where u^W is the unique solution of (2.2) given in (i) of Proposition 1, and where B_R is endowed with the topology of $C([0, T_0])$.

Remark 2.5. *In the proof of Proposition 2, a uniform estimate on u^W, for $W \in B_R$ is required on a fixed (possibly small) time interval. This is the reason why Theorem 3 does not cover the supercritical cases $\sigma \geq 2/d$, as e.g. $d = 3$ and $\sigma = 1$, although the latter is an interesting physical case (see* [16]*). The reader may refer to Remark 5.2 for details.*

Remark 2.6. *The occurrence of blow-up with positive probability for a certain initial data was proved for the equation (1.5) with $V = K = |x|^2$, $\lambda = 1$ and $\sigma \geq 2/d$ (see* [12]*) establishing the associated virial identity. This identity does not give any information about the exact time of blow-up even in the deterministic case. Seeing the influence by the noise on the blow-up time would be an interesting question, but this problem is under investigation by numerical simulations.*

3. Linear problem, phase flow, construction of the propagator

In this section, we consider the linear equation (1.3) and follow closely Section 2 of [28]. Our aim is the construction of the propagator of (1.3), and the investigation of some properties of its integral kernel. We only give the outline of the arguments in this section since most of them follow those of Yajima [28]. Some brief proofs corresponding to the statements in this section will be found in the appendix. In all the section, $T_0 > 0$ is fixed.

We first study the small time behaviour of the phase flow generated by the Hamiltonian

$$H_0(t, x, \xi) = \frac{1}{2}\Big(\xi - A(t,x)\Big)^2,$$

where we recall that $A(t,x) = \frac{1}{2}(\nabla V(x)t + \sigma_0 \nabla K(x) W(t))$, i.e. $A_j(t,x) = x_j(\nu_j t + \sigma_0 \kappa_j W(t))$. In all what follows, we denote by N (resp. Γ) the diagonal $d \times d$ matrix such that $\frac{1}{2}\nabla V(x) = Nx$ (resp. $\frac{\sigma_0}{2}\nabla K(x) = \Gamma x$) for any $x \in \mathbb{R}^d$. Then, the Hamilton's equations read as follows:

$$\begin{cases} \dot{x}(t) & = \partial_\xi H_0(t,x,\xi) = \xi(t) - tNx(t) - W(t)\Gamma x(t) \\ \dot{\xi}(t) & = -\partial_x H_0(t,x,\xi) = (tN + W(t)\Gamma)(\xi(t) - tNx(t) - W(t)\Gamma x(t)) \end{cases} \tag{3.1}$$

with $(x(s), \xi(s)) = (y, \eta) \in \mathbb{R}^{2d}$. We assume that $\omega \in \Omega$ is such that $W(\cdot, \omega) \in C^\alpha([0, T_0])$, where $0 < \alpha < 1/2$ is fixed. It follows from a fixed point argument in $C([s, s+T]; \mathbb{R}^{2d})$, with T sufficiently small, and the fact that the system (3.1) is linear in (x, ξ), that for any $s \in [0, T_0]$, there exists a unique solution of the above system, in $C([0, T_0]; \mathbb{R}^{2d})$, with $(x(s), \xi(s)) = (y, \eta)$, denoted by $(x(., s, y, \eta), \xi(., s, y, \eta))$, and verifying

$$\sup_{t \in [0, T_0]} (|x(t)| + |\xi(t)|) \leq C_{\omega, T_0}(1 + |y| + |\eta|). \tag{3.2}$$

Remark 3.1. *It will be useful to remark here that (x, ξ) is linear with respect to y and η because the system (3.1) is linear. Moreover, it is immediate that $(x, \xi) \in C^1([0, T_0]; \mathbb{R}^{2d})$.*

We also set for $t \neq s$,

$$\tilde{x}(t, s, y, \eta) = x\Big(t, s, y, \frac{\eta}{t-s}\Big), \quad \tilde{\xi}(t, s, y, \eta) = (t-s)\xi\Big(t, s, y, \frac{\eta}{t-s}\Big) \tag{3.3}$$

and $\tilde{x}(s, s, y, \eta) = y + \eta$, $\tilde{\xi}(s, s, y, \eta) = \eta$.

We then have the following properties concerning $\tilde{x}$ and $\tilde{\xi}$. We omit the proofs, except the C^1 regularity in time for $|t - s|$ small (Proposition 5 below), which is given in the appendix.

Proposition 3. *For $t, s \in [0, T_0]$, $t \neq s$, let $(\tilde{x}(., s, y, \eta), \tilde{\xi}(., s, y, \eta))$ be defined by (3.3).*

(1) *For any multi-indices α and β, $\partial_y^\alpha \partial_\eta^\beta \tilde{x}(t, s, y, \eta)$ and $\partial_y^\alpha \partial_\eta^\beta \tilde{\xi}(t, s, y, \eta)$ are C^1 in (t, s, y, η) for $t, s \in [0, T_0]$, $t \neq s$, and $(y, \eta) \in \mathbb{R}^{2d}$. Moreover, for $1 \leq j, l \leq d$,*

$$\Big|\partial_y^\alpha \partial_\eta^\beta \Big\{\frac{\partial \tilde{x}_j}{\partial y_l} - \delta_{jl}\Big\}\Big| + \Big|\partial_y^\alpha \partial_\eta^\beta \Big\{\frac{\partial \tilde{x}_j}{\partial \eta_l} - \delta_{jl}\Big\}\Big|$$
$$+\Big|\partial_y^\alpha \partial_\eta^\beta \Big\{\frac{\partial \tilde{\xi}_j}{\partial y_l}\Big\}\Big| + \Big|\partial_y^\alpha \partial_\eta^\beta \Big\{\frac{\partial \tilde{\xi}_j}{\partial \eta_l} - \delta_{jl}\Big\}\Big| \leq C_{\alpha, \beta, \omega, T_0}|t - s|.$$

(2) *There exists a positive number $T_\omega > 0$ such that, for $t, s \in [0, T_0]$ with $|t - s| \leq T_\omega$, the mappings $(y, \eta) \mapsto (x, \eta) = (\tilde{x}(t, s, y, \eta), \eta)$, $(y, \eta) \mapsto (y, \xi) = (y, \tilde{\xi}(t, s, y, \eta))$ and*

$$(y, \eta) \mapsto (y, x) = (y, \tilde{x}(t, s, y, \eta)) \tag{3.4}$$

are global diffeomorphisms on $\mathbb{R}^d \times \mathbb{R}^d$.

(3) *Let $(y, \tilde{\eta}(t, s, y, x))$ be the inverse of (3.4) and $\eta(t, s, y, x) = \tilde{\eta}(t, s, y, x)/(t - s)$. Then*

$$\tau \mapsto (q(\tau), \xi(\tau)) = (\tilde{x}(\tau, s, y, \tilde{\eta}(t, s, y, x)), \tilde{\xi}(\tau, s, y, \tilde{\eta}(t, s, y, x))) \tag{3.5}$$

is the unique solution of (3.1) such that $q(s) = y$ and $q(t) = x$.

We define, for $|t - s| \leq T_\omega$ and for $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$, the action integral of the path $(q(\tau), v(\tau))$ given by (3.5) and $v(\tau) = \frac{dq}{d\tau} = \xi(\tau) - A(\tau, q(\tau))$ as follows:

$$S(t, s, x, y) = \int_s^t L(\tau, q(\tau), v(\tau))d\tau, \tag{3.6}$$

where $L(t, q, v)$ is the Lagrangian associated to $H_0(t, q, \xi)$, that is,

$$L(t, q, v) = v \cdot \xi - H_0(t, q, \xi) = \frac{1}{2}(\xi^2 - A^2(t, q)) = \frac{v^2}{2} + v \cdot (tN + W(t)\Gamma)q. \tag{3.7}$$

For $t \neq s$, it is easily seen that $S(t, s, x, y)$ is C^1 in (t, s, x, y), and that $S(t, s, x, y)$ is a generating function of the map $(y, \eta) \mapsto (x(t, s, y, \eta), \xi(t, s, y, \eta))$. More precisely,

Proposition 4. *For $t, s \in [0, T_0]$, There exists $T_\omega > 0$ such that for any multi-indices α and β, $\partial_x^\alpha \partial_y^\beta S(t,s,x,y)$ is C^1 in (t,s,x,y) for $0 < |t-s| \le T_\omega$ and $(x,y) \in \mathbb{R}^{2d}$, moreover*

$$(\nabla_x S)(t,s,x(t,s,y,\eta),y) = \xi(t,s,y,\eta), \tag{3.8}$$

$$(\nabla_y S)(t,s,x(t,s,y,\eta),y) = -\eta, \tag{3.9}$$

$$(\partial_t S)(t,s,x,y) + (1/2)\big((\nabla_x S)(t,s,x,y) - A(t,x)\big)^2 = 0, \tag{3.10}$$

$$(\partial_s S)(t,s,x,y) - (1/2)\big((\nabla_y S)(t,s,x,y) + A(s,y)\big)^2 = 0, \tag{3.11}$$

$$\Big|\partial_x^\alpha \partial_y^\beta \Big\{S(t,s,x,y) - \frac{|x-y|^2}{2(t-s)}\Big\}\Big| \le C_{\alpha,\beta,\omega,T_0}, \quad |\alpha+\beta| = 2; \tag{3.12}$$

finally, $S(t,s,x,y)$ is quadratic in (x,y), that is, $\partial_x^\alpha \partial_y^\beta S(t,s,x,y) = 0$ for $|\alpha+\beta| \ge 3$.

The proof of (3.12) may be performed as in Yajima [28], introducing $\tilde{S}(t,s,x,y) = (t-s)S(t,s,x,y)$, using the fact that $\tilde{S}(t,s,x,y)$ is a generating function of the mapping $(y,\eta) \mapsto (\tilde{x}, \tilde{\xi})$, that is,

$$\begin{aligned}(\partial_x \tilde{S})(t,s,x,y) &= \tilde{\xi}(t,s,y,\tilde{\eta}(t,s,y,x)), &(3.13)\\ (\partial_y \tilde{S})(t,s,x,y) &= -\tilde{\eta}(t,s,y,x), &(3.14)\end{aligned}$$

and Proposition 3.

We will prove that the definition of $\tilde{S}(t,s,x,y)$ eliminates the singularity at $t = s$ in $S(t,s,x,y)$, and that the following smoothness properties hold.

Proposition 5. *For any multi-indices γ and β, with $|\gamma+\beta| \le 2$, $\partial_x^\gamma \partial_y^\beta \tilde{S}(t,s,x,y)$ is C^1 in (t,s,x,y) for $|t-s| \le T_\omega$ and $(x,y) \in \mathbb{R}^{2d}$. Moreover,*

$$\begin{aligned}&\Big|\partial_x^\gamma \partial_y^\beta \Big(\tilde{S}(t,s,x,y) - \frac{1}{2}(x-y)^2 - \frac{1}{2}(t-s)(x-y)\cdot(sN + W(s)\Gamma)(x+y)\Big)\Big| &(3.15)\\ \le\ & C_{\omega,T_0}|t-s|^{1+\alpha}(1+|x|+|y|)^{2-|\gamma+\beta|},\end{aligned}$$

$$\begin{aligned}&\Big|\partial_x^\gamma \partial_y^\beta \Big((\partial_t \tilde{S})(t,s,x,y) - \frac{1}{2}(x-y)\cdot(sN + W(s)\Gamma)(x+y)\Big)\Big| &(3.16)\\ \le\ & C_{\omega,T_0}|t-s|^{\alpha}(1+|x|+|y|)^{2-|\gamma+\beta|},\end{aligned}$$

$$\begin{aligned}&\Big|\partial_x^\gamma \partial_y^\beta \Big((\partial_s \tilde{S})(t,s,x,y) + \frac{1}{2}(x-y)\cdot(sN + W(s)\Gamma)(x+y)\Big)\Big| &(3.17)\\ \le\ & C_{\omega,T_0}|t-s|^{\alpha}(1+|x|+|y|)^{2-|\gamma+\beta|},\end{aligned}$$

where we recall that $0 < \alpha < 1/2$ is such that $W(\cdot,\omega) \in C^\alpha([0,T_0])$.

Proof. see Appendix. □

Still following the idea in [28], we set, for $|t-s| \le T_\omega$,

$$R(t,s) = -(\Delta_x S)(t,s) + \frac{d}{t-s} + s\mathrm{Tr}(N) + W(s)\mathrm{Tr}(\Gamma), \tag{3.18}$$

and

$$a(t,s) = \exp\Big(\frac{1}{2}\int_s^t R(\tau,s)d\tau\Big). \tag{3.19}$$

Note that in our case, $R(t,s)$ and $a(t,s)$ do not depend on (x,y) because $S(t,s,x,y)$ is quadratic in (x,y). We easily deduce from Propositions 4 and 5 that $R(t,s)$ is a continuous function of $(t,s) \in [0,T_0]^2$ with $|t-s| \le T_\omega$, and $a(t,s)$ is a real valued C^1 function of $(t,s) \in [0,T_0]^2$ with $|t-s| \le T_\omega$ verifying

$$|a(t,s) - 1| \le C_{\omega,T_0}|t-s|. \tag{3.20}$$

Next, we define, for $t,s \in [0,T_0]$ and $0 < |t-s| \le T_\omega$, the oscillatory integral operator:

$$I(t,s,a)f(x) = (2\pi i(t-s))^{-d/2} a(t,s) \int_{\mathbb{R}^d} e^{iS(t,s,x,y)} f(y)dy, \quad f \in C_0^\infty(\mathbb{R}^d). \tag{3.21}$$

We now list the properties of the oscillatory integral operator I that will allow us to define the propagator U^ω and to conclude the proof of Theorem 1.

Proposition 6. *Let $(t,s) \in [0,T_0]^2$ with $0 < |t-s| \le T_\omega$. Let $I(t,s,a)$ be the oscillatory integral operator defined in (3.21), and $a(t,s)$ be the function defined in (3.19). Then the following properties hold.*

(1) *The adjoint operator $I(t,s,a)^*$ of $I(t,s,a)$ satisfies $I(t,s,a)^* = I(s,t,\tilde{a})$, with $\tilde{a}(t,s) = a(s,t)$.*
(2) *There exist bounded real valued functions $a_{jk,lm}(t,s)$, with $l,m = 1,2$ and $1 \le j,k \le d$, such that*

$$\begin{aligned} x_j I(t,s,a) &= I(t,s,a)x_j - (t-s)I(t,s,a)(i\partial_{x_j}) \\ &\quad + (t-s)\sum_{k=1}^d \{I(t,s,a_{jk,11})x_k + I(t,s,a_{jk,12})(i\partial_{x_k})\}, \end{aligned} \tag{3.22}$$

$$i\partial_{x_j} I(t,s,a) = I(t,s,a)i\partial_{x_j} + \sum_{k=1}^d \{I(t,s,a_{jk,21})x_k + (t-s)I(t,s,a_{jk,22})(i\partial_{x_k})\}. \tag{3.23}$$

(3) *For any $k \in \mathbb{N}$, $I(t,s,a)$ is a continuous operator in $\Sigma(k)$, and*

$$|I(t,s,a)f|_{\Sigma(k)} \le C_{\omega,T_0,k}|f|_{\Sigma(k)}.$$

(4) *For any $s \in [0,T_0]$ and $f \in L^2(\mathbb{R}^d)$, we have $\lim_{t\to s}|I(t,s,a)f - f|_{L^2} = 0$.*

Proof. see Appendix.

It is not difficult, using (3.10), to prove that the operator $I(t,s,a)$ satisfies, for any $f \in C_0^\infty(\mathbb{R}^d)$,

$$\Big(i\partial_t + \frac{1}{2}\Big(\nabla - iA(t,x)\Big)^2\Big)I(t,s,a)f(x) = 0. \tag{3.24}$$

Thus, setting $U^\omega(t,s) = I(t,s,a)$, Proposition 6 implies that U^ω is a unitary propagator for equation (1.3) (see the proof of Theorem 3 in [28]), that $U^\omega(t,s)f$ satisfies (1.3) in $L^2(\mathbb{R}^d)$ if $f \in \Sigma(2)$, and that $U^\omega(t,s)f \in C([s,s+T_\omega];L^2)$ if $f \in L^2(\mathbb{R}^d)$. These arguments prove Theorem 1.

Remark 3.2. *Once $U^\omega(t,s)$ is defined for all $(t,s) \in [0,T_0]^2$ (see Remark 2.3) one obtains that $(u(t))_{t\ge s} = U^\cdot(t,s)u_s$ is adapted to $(\mathcal{F}_t)_{t\ge s}$, provided $u_s \in L^2(\Omega,\mathcal{F}_s,L^2(\mathbb{R}^d))$. Indeed, it easily follows from (3.1) and (2) of Proposition 3 that $\tilde{x}(t,s,y,\eta)$, $\tilde{\xi}(t,s,y,\eta)$ and $\tilde{\eta}(t,s,y,x)$ are $\mathcal{F}_t$-measurable, hence so is $S(t,s,x,y)$ by (3.6); on the other hand, (3.19) and (3.18) show that $a(t,s)$ is $\mathcal{F}_t$-measurable.*

Remark 3.3. *It may easily be seen that all the constants C_{ω,T_0} appearing in this section are uniform in the $C^\alpha([0,T_0])$-norm of $W(.,\omega)$, that is, these constants depend only on R when $W(.,\omega)$ belongs to the ball of radius R in $C^\alpha([0,T_0])$. This remark will be useful in the proof of Theorem 3.*

4. Strichartz estimates and nonlinear evolution

We prove Proposition 1 and Theorem 2 in this section. For this purpose we first establish the Strichartz estimates. We remark that the expression (3.21) gives $L^1 \to L^\infty$ estimate of the propagator and the Riesz-Thorin interpolation theorem implies the following lemma 4.1. $T_\omega > 0$ will always be assumed sufficiently small so that the previous arguments in Section 3 are satisfied.

Lemma 4.1. *Let $2 \le p \le \infty$ and $t, s \in [0, T_0]$, with $|t-s| \le T_\omega$. Let $U^\omega(t,s)$ be the unique propagator of (1.3) established in Section 3. For any $f \in L^{p'}(\mathbb{R}^d)$, the following estimate holds.*

$$|U^\omega(t,s)f|_{L^p(\mathbb{R}^d)} \le \frac{C_{\omega,T_0}}{|t-s|^{d(1/2-1/p)}} |f|_{L^{p'}(\mathbb{R}^d)},$$

where p' is the conjugate number of p given by $\frac{1}{p} + \frac{1}{p'} = 1$.

Recall that a pair (q,r) is said to be admissible if $\frac{2}{q} = d\Big(\frac{1}{2} - \frac{1}{r}\Big)$ and $2 \le r < \frac{2d}{d-2}$ ($2 \le r \le \infty$ if $d=1$, $2 \le r < \infty$ if $d=2$).

Proposition 7. *(Strichartz estimates) Let (q,r) be an admissible pair. There is a constant C_{ω,T_0} such that for any $s \in [0,T_0]$ and any $u_s \in L^2(\mathbb{R}^d)$,*

$$|U^\omega(\cdot,s)u_s|_{L^q(s,s+T_\omega\wedge T_0;L^r)} \le C_{\omega,T_0}|u_s|_{L^2}. \tag{4.1}$$

If (γ,ρ) is an admissible pair and $f \in L^{\gamma'}(s,s+T_\omega;L^{\rho'}(\mathbb{R}^d))$, then $\Lambda^\omega(t,s)f$ defined as

$$\Lambda^\omega(t,s)f = \int_s^t U^\omega(t,\tau)f(\tau)d\tau, \quad t \in [s, s+T_\omega \wedge T_0]$$

belongs to $L^q(s,s+T_\omega\wedge T_0;L^r(\mathbb{R}^d)) \cap C([s,s+T_\omega\wedge T_0];L^2(\mathbb{R}^d))$. Furthermore, there exists a constant C_{ω,T_0} such that, for every $f \in L^{\gamma'}(s,s+T_\omega;L^{\rho'}(\mathbb{R}^d))$,

$$|\Lambda^\omega(\cdot,s)f|_{L^q(s,s+T_\omega\wedge T_0;L^r)} \le C_{\omega,T_0}|f|_{L^{\gamma'}(s,s+T_\omega;L^{\rho'})}. \tag{4.2}$$

Proof of Proposition 7. Here we give only the ideas of the proof since we can prove this proposition similarly to Theorem 2.3.3 in [7]. First we obtain the estimate, for any admissible pair (q,r) and $f \in L^{q'}(s,s+T_\omega;L^{r'}(\mathbb{R}^d))$,

$$|\Lambda^\omega(\cdot,s)f|_{L^q(s,s+T_\omega\wedge T_0;L^r)} \le C_{\omega,T_0}|f|_{L^{q'}(s,s+T_\omega;L^{r'})} \tag{4.3}$$

using the Riesz potential inequalities and Lemma 4.1. All the other inequalities are obtained using duality and interpolation estimates (see [7]), e.g., for any admissible pair (q,r) and $f \in L^{q'}(s,s+T_\omega;L^{r'}(\mathbb{R}^d))$, knowing (4.3),

$$\begin{aligned}
|\Lambda^\omega(t,s)f|_{L^2}^2 &= \Re\int_s^t\int_s^t \langle U^\omega(t,\sigma)f(\sigma), U^\omega(t,\theta)f(\theta)\rangle d\sigma d\theta \\
&= \Re\int_s^t\int_s^t \langle f(\sigma), U^\omega(t,\sigma)^*U^\omega(t,\theta)f(\theta)\rangle d\sigma d\theta \\
&= \Re\int_s^t \langle f(\sigma), \int_s^t I(\sigma,\theta,c)f(\theta)d\theta\rangle d\sigma \\
&\le |f|_{L^{q'}(s,t;L^{r'})}^2
\end{aligned}$$

by duality for any $t \in [s, s+T_\omega \wedge T_0]$ and for some function $c = c(t,s,x,y,\omega)$. Note that $c(t,s,\cdot,\cdot,\omega)$ in the third equality is a continuous and bounded function on $\mathbb{R}^d \times \mathbb{R}^d$ for $(t,s) \in [0,T_0]^2$ with $0 < |t-s| \le T_\omega$, whose existence is ensured by Proposition 6 and Lemma 3.1 of [28]. $\square$

Once established the Strichartz estimate, we are in position to prove the local existence of solutions.

Proof of Proposition 1. Let $0 < T \le T_0 \wedge T_\omega$ where T_0 is fixed, and put $I = [0,T]$. As in [17, 27], the local existence in L^2 is proved by a fixed point method in $B^X_{R_0} = \{v \in X_T, |v|_{X_T} \le R_0\}$, $R_0 > 0$, where

$$X_T = L^\infty(I; L^2) \cap L^r(I; L^{2\sigma+2}),$$

with the metric $|v|_{X_T} = |v|_{L^\infty(I;L^2)} + |v|_{L^r(I;L^{2\sigma+2})}$ and r is such that $(r, 2\sigma+2)$ is an admissible pair. Let $u_0 \in L^2(\mathbb{R}^d)$. We define the mapping $\mathcal{T}^\omega$ by

$$(\mathcal{T}^\omega u)(t) = U^\omega(t,0)u_0 - i\lambda \int_0^t U^\omega(t,\tau)(|u|^{2\sigma}u(\tau))d\tau = U^\omega(t,0)u_0 - i\lambda\Lambda^\omega(t,0)(|u|^{2\sigma}u). \tag{4.4}$$

For $u, v \in B^X_{R_0}$, the estimates

$$|\mathcal{T}^\omega u|_{X_T} \le C_{\omega,T_0}|u_0|_{L^2} + C_{\omega,T_0}T^\gamma |u|^{2\sigma+1}_{X_T}$$

and

$$\begin{aligned} |\mathcal{T}^\omega u - \mathcal{T}^\omega v|_{X_T} &\le C_{\omega,T_0}T^\gamma(|u|^{2\sigma}_{L^r_I L^{2\sigma+2}_x} + |v|^{2\sigma}_{L^r_I L^{2\sigma+2}_x})|u-v|_{L^r_I L^{2\sigma+2}_x} &\quad (4.5)\\ &\le C_{\omega,T_0}T^\gamma {R_0}^{2\sigma}|u-v|_{X_T} &\quad (4.6)\end{aligned}$$

hold with $\gamma = 1 - \frac{2\sigma+2}{r}$ which is positive if $\sigma < 2/d$. Taking $R_0^\omega = 2C_{\omega,T_0}|u_0|_{L^2}$, and choosing T sufficiently small, $\mathcal{T}^\omega$ maps $B^X_{R_0^\omega}$ into itself, and is a contraction mapping. T depends only on $|u_0|_{L^2}, \omega$ and T_0.

Before proving the conservation of the L^2 norm, let us prove the local existence of continuous solutions with values in Σ. We define the space,

$$Y_T = \{v \in X_T,\ xv, \nabla v \in L^\infty(I; L^2) \cap L^r(I; L^{2\sigma+2})\}.$$

Note that the ball $B^Y_R = \{v \in Y_T, |v|_{Y_T} \le R\}$ is closed for the norm $|\cdot|_{X_T}$. Let $u_0 \in \Sigma$. We prove that $\mathcal{T}^\omega$ defined above is a contraction mapping in the ball B^Y_R, for a well chosen R. We remark here that ∂_{x_j} and the multiplication by x_j do not commute with $U^\omega(t,0)$, but by virtue of (2) of Proposition 6, we have the following estimates.

Lemma 4.2. *For any $f \in \Sigma$, $U^\omega(t,0)f \in C(I,\Sigma) \cap Y_T$. Moreover,*

$$|U^\omega(\cdot,0)f|_{Y_T} \le C_1|f|_\Sigma, \quad |\Lambda_\omega(\cdot,0)f|_{Y_T} \le C_2|f|_{Y'_T}$$

with constants C_1 and C_2 independent of T (but depending on ω and T_0), where

$$Y'_T = \{v \text{ such that } v, xv, \nabla v \in L^1(I; L^2) + L^{r'}(I; L^{\frac{2\sigma+2}{2\sigma+1}})\}.$$

Proof. see Lemma 3.3 of [9].

Let $u \in B^Y_R$. With the help of the above lemma, we can show, in addition to (4.6), that

$$|\mathcal{T}^\omega u|_{Y_T} \le C_1|u_0|_\Sigma + C_3 T^\gamma |u|^{2\sigma}_{L^r_I L^{2\sigma+2}_x}|u|_{Y_T}.$$

Since, from the proof of (i), we have $|u|_{L^r_I L^{2\sigma+2}_x} \le 2C_{\omega,T_0}|u_0|_{L^2}$, we may choose $R = 2C_1|u_0|_\Sigma$ and $T > 0$ sufficiently small so that $\mathcal{T}^\omega$ is a contraction mapping from B^Y_R into itself (for the X_T-norm). It may be seen that T depends only on $|u_0|_{L^2}$, ω and T_0, and thus the solution is in Y_T as long as it exists in X_T.

Let us now prove the conservation of the L^2 norm with the use of a regularization procedure. We first assume that the initial data u_0 is in Σ. Consider a function $\rho \in C_0^\infty(\mathbb{R}^d)$ satisfying $\rho \ge 0$ and

$\int_{\mathbb{R}^d} \rho(x)dx = 1$. Let $\rho_\varepsilon(x) = \varepsilon^{-d}\rho(\frac{x}{\varepsilon})$. We regularize the equation (2.2) by this molifier ρ_ε. Since $A(x,t)$ is linear in x, the regularized equation is written as follows.

$$i\partial_t(\rho_\varepsilon * u) = -\frac{1}{2}\Big(\nabla - iA(t,x)\Big)^2(\rho * u) + \lambda\rho_\varepsilon * (|u|^{2\sigma}u) - \frac{1}{2}(A^2\rho_\varepsilon) * u - i(A\rho_\varepsilon) * ((\nabla - iA)u).$$

Thus, the evolution of L^2 norm for this equation is, for any $t \in [0,T]$

$$\begin{aligned} |\rho_\varepsilon * u(t)|_{L^2}^2 &= |\rho_\varepsilon * u_0|_{L^2}^2 + 2\lambda\Im\int_0^t \langle \rho_\varepsilon * (|u|^{2\sigma}u), \rho_\varepsilon * u\rangle ds \\ &\quad -\Im\int_0^t \langle (A^2\rho_\varepsilon) * u, \rho_\varepsilon * u\rangle ds - 2\Re\int_0^t \langle (A\rho_\varepsilon) * ((\nabla - iA)u), \rho_\varepsilon * u\rangle ds. \end{aligned}$$

The last two terms in the right hand side tend to zero as ε goes to 0, since $(A^k\rho_\varepsilon) * v$ with $k = 1,2$ tends to zero in $L^2(0,t;L^2)$ for any $v \in L^2(0,t,L^2)$, and $\rho_\varepsilon * u$ converges to u in $L^2(0,t;L^2)$. We apply the convergences $\rho_\varepsilon * u \to u$ in $L^r(0,t;L^{2\sigma+2})$ and $\rho_\varepsilon * (|u|^{2\sigma}u) \to |u|^{2\sigma}u$ in $L^{r'}(0,t;L^{(2\sigma+2)/(2\sigma+1)})$ to the second term in the right hand side which also vanishes as ε goes to zero. On the other hand, by the fact $\rho_\varepsilon * u(t)$ converges to $u(t)$ and $\rho_\varepsilon * u_0$ converges to u_0 in L^2 as ε tends to zero, we obtain the L^2 conservation for the solution of (2.2) for any $t \in [0,T]$, if the initial data u_0 is in Σ. THis is obviously still true for any initial data in L^2 by regularization. Thanks to the conservation of the L^2 norm, we get the global existence of solutions in L^2, and thus also in Y_T when the initial data is in Σ. Continuous dependence on the initial data in L^2 is shown similarly to Theorem 4.1 in [9]. This completes part (i) and (ii) of the proof of Proposition 1.

Lastly, we give the arguments for proving (iii) of Proposition 1. In that case, we estimate the non-linearity as $||u|^{2\sigma}u|_{Y_T'} \le CT^{1-\theta}|u|_{Y_T}^{2\sigma+1}$ for any $u \in Y_T$ and $\theta = \frac{d\sigma}{2(\sigma+1)} < 1$. Then, if $u_0 \in \Sigma$, for any $u, v \in B_R^Y$, we have

$$|\mathcal{T}^\omega u|_{Y_T} \le C_1|u_0|_\Sigma + C_2T^{1-\theta}|u|_{Y_T}^{2\sigma+1},$$

and

$$|\mathcal{T}^\omega u - \mathcal{T}^\omega v|_{X_T} \le CT^{1-\theta}(|u|_{Y_T}^{2\sigma} + |v|_{Y_T}^{2\sigma})|u - v|_{X_T},$$

which implies that $\mathcal{T}^\omega$ is a contraction in B_R^Y with $R = 2C_1|u_0|_\Sigma$, for sufficiently small $T > 0$. This allows us to show the local existence and the blow-up alternative in Σ (see [9]). Note that by virtue of Proposition 7 and (4) of Proposition 6, in each case, $\mathcal{T}^\omega u$ belongs to $C(I;L^2)$ or $C(I;\Sigma)$. The adaptivity of u results from the adaptivity of U^ω (see Remark 3.2), the fact that u is obtained by a fixed point procedure, and the use of a cut-off argument (see e.g. [10] or [11]). In the supercritical case (iii), the cut-off argument has to be performed for a fixed t, in $L^{2\sigma+2}$ norm. Note that the adaptivity of u implies that T^* is a stopping time. □

Proof of Theorem 2. We note that if $u \in C([0,T_0];L^2(\mathbb{R}^d))$ then ψ given by (1.2) is also in $C([0,T_0];L^2(\mathbb{R}^d))$; moreover, if u is a solution of Eq. (2.2), then the Itô formula implies that ψ solves Eq. (1.5) in $C([0,T_0];\Sigma(-2))$. Since in addition, $u \in C([0,T_0];\Sigma)$ implies $\psi \in C([0,T_0];\Sigma)$, it is easily seen that the results of (i) and (ii) in Proposition 1 imply (i) and (ii) of Theorem 2. Concerning the local existence in Σ, (iii), we define for $R > 0$, $\tau_R = \inf\{t \ge 0, |u(\cdot)|_{L^\infty(0,t;\Sigma)} \ge R\}$ where u is the solution obtained in (iii) of Proposition 1, with $u_0 = \psi_0$. Since $\{u(t)\}_{t\ge0}$ is adapted to $(\mathcal{F}_t)_{t\ge0}$, τ_R is an increasing stopping time. We then set $\tau^* = \lim_{R\to+\infty}\tau_R$. On the other hand, by the deterministic theory, we know that there exists a maximal time $T^* = T^*_{\omega,u_0} > 0$ such that the following alternative holds; $T^* = +\infty$ or $\lim_{t\uparrow T^*}|u(t)|_\Sigma = +\infty$ if $T* < +\infty$. If $T^* = +\infty$, u exists globally, so ψ is global, too. If $T^* < +\infty$, the definition of τ_R implies $\tau^* = T^*$. Part (iv) follows from the same argument as in (i) of Theorem 3 of [12], combined with some ideas in [6]. Using the Itô formula, the evolution of the Hamiltonian H given by (2.4) of the solution of (1.5) is found to be, for any stopping time $\tau \leq \tau_R \wedge T$:

$$H(\psi(\tau)) = H(\psi_0) + \frac{1}{4}\int_0^\tau \int_{\mathbb{R}^d} |\Gamma x|^2 |\psi(x)|^2 dx dt - \frac{1}{2}\mathrm{Im}\int_0^\tau \int_{\mathbb{R}^d} \Gamma x \cdot \nabla\psi(x)\bar{\psi}(x) dx dW, \quad \text{a.s;} \tag{4.7}$$

on the other hand, again by the Itô formula,

$$|x\psi(\tau)|_{L^2}^2 = |x\psi_0|_{L^2}^2 + 2\mathrm{Im}\int_0^\tau \int_{\mathbb{R}^d} x\bar{\psi}\cdot\nabla\psi dx dt, \quad \text{for any } \tau \leq \tau_R \wedge T, \quad \text{a.s.} \tag{4.8}$$

Now, assume that $\lambda = +1$; one easily get form (4.7) that for any $R > 0$,

$$\begin{aligned} \mathbb{E}\Big(\sup_{t\in[0,\tau_R\wedge T]} |\nabla\psi|_{L^2}^2\Big) &\leq |H(\psi_0)| + C_{|N|,d}\mathbb{E}\Big(\sup_{t\in[0,\tau_R\wedge T]} |x\psi(t)|_{L^2}^2\Big) \\ &\quad + C_{|\Gamma|,d,\sigma_0}\mathbb{E}\Big(\sup_{t\in[0,\tau_R\wedge T]} \int_0^t |x\psi(s)|_{L^2}^2 ds\Big) \\ &\quad + C_{\sigma_0}\mathbb{E}\Big(\sup_{t\in[0,\tau_R\wedge T]} \Big|\int_0^t \int_{\mathbb{R}^d} \nabla K\cdot\nabla\psi\bar{\psi} dx dW\Big|\Big) \end{aligned} \tag{4.9}$$

The last term of the right hand side above is estimated thanks to Theorem 3.14 in [8], and is then majorized by

$$\begin{aligned} 3C_{\sigma_0}\mathbb{E}\Big[\Big(\int_0^{T\wedge\tau_R} \Big|\int_{\mathbb{R}^d} \nabla K\cdot\nabla\psi\bar{\psi} dx\Big|^2 ds\Big)^{1/2}\Big] &\leq 3C_{\sigma_0}T\mathbb{E}\Big(\sup_{t\in[0,\tau_R\wedge T]} |\nabla\psi(t)|_{L^2}|x\psi(t)|_{L^2}\Big) \\ &\leq \frac{1}{2}\mathbb{E}\Big(\sup_{t\in[0,\tau_R\wedge T]} |\nabla\psi(t)|_{L^2}^2\Big) + C_{\sigma_0,T}\mathbb{E}\Big(\sup_{t\in[0,\tau_R\wedge T]} |x\psi(t)|_{L^2}^2\Big). \end{aligned}$$

Plugging this estimate into (4.9), one gets

$$\begin{aligned} \mathbb{E}\Big(\sup_{t\in[0,\tau_R\wedge T]} |\nabla\psi|_{L^2}^2\Big) &\leq 2|H(\psi_0)| + C_{|N|,d,T,\sigma_0}\mathbb{E}\Big(\sup_{t\in[0,\tau_R\wedge T]} |x\psi(t)|_{L^2}^2\Big) \\ &\quad + C_{|\Gamma|,d,\sigma_0}\mathbb{E}\Big(\sup_{t\in[0,\tau_R\wedge T]} \int_0^t |x\psi(s)|_{L^2}^2 ds\Big). \end{aligned} \tag{4.10}$$

On the other hand, by (4.8), one has

$$\begin{aligned} \mathbb{E}\Big(\sup_{t\in[0,\tau_R\wedge T]} |x\psi(t)|_{L^2}^2\Big) &\leq |x\psi_0|_{L^2}^2 \\ &\quad + \mathbb{E}\Big(\int_0^T \mathbb{1}_{[0,\tau_R\wedge T]}(|\nabla\psi(t)|_{L^2}^2 + |x\psi(t)|_{L^2}^2) dt\Big). \end{aligned} \tag{4.11}$$

Hence, combining (4.10) and (4.11),

$$\begin{aligned} &\mathbb{E}\Big(\sup_{t\in[0,\tau_R\wedge T]} |\nabla\psi(t)|_{L^2}^2 + |x\psi(t)|_{L^2}^2\Big) \\ &\leq C_{|N|,d,T,\sigma_0,|\psi_0|_\Sigma} + C_{|N|,d,T,\sigma_0,|\Gamma|}\mathbb{E}\Big(\int_0^T \mathbb{1}_{[0,\tau_R\wedge T]}(|\nabla\psi(t)|_{L^2}^2 + |x\psi(t)|_{L^2}^2) dt\Big), \end{aligned}$$

and one concludes using Gronwall's lemma that

$$\mathbb{E}\Big(\sup_{t\in[0,\tau_R\wedge T]} |\nabla\psi(t)|_{L^2}^2 + |x\psi(t)|_{L^2}^2\Big) \leq C_{|\psi_0|_\Sigma,|N|,d,T,\sigma_0,|\Gamma|}.$$

This latter estimate implies $\tau^* = +\infty$, a.s. □

Remark 4.1. *Note that the estimates proving the local existence of solutions in X_T or Y_T in the proof of Proposition 1 are still available in the presence of a damping term $-i\gamma u$, with $\gamma > 0$, on the right hand side of Eq.(2.2). In this case the L^2 norm is not preserved but is decreasing. Moreover, in the proof of (iv) of Theorem 2, (4.7) may be replaced by the equality (3.3) of* [12] *and (4.8) by*

$$|x\psi(\tau)|_{L^2}^2 = |x\psi_0|_{L^2}^2 + 2\mathrm{Im}\int_0^\tau \int_{\mathbb{R}^d} x\bar{\psi}\cdot\nabla\psi dxdt - 2\gamma\int_0^\tau |x\psi(t)|_{L^2}^2 dt$$

if $\gamma > 0$ in Eq.(1.1), and this leads to the same conclusion.

5. Continuous dependence on the Brownian path and convergence

This section is devoted to the proof of Proposition 2, i.e. the continuous dependence of solutions on the Brownian paths, and of Theorem 3.

We begin with introducing the following proposition which is a consequence of the properties of the propagator $U^\omega(t,s)$ that we studied in Section 3. We have already seen that for $f \in C_0^\infty(\mathbb{R}^d)$, $U^\omega(\cdot,s)f$ is a strong solution of (1.3), and is a function of $W(\cdot,\omega)$, provided $\omega \in \tilde{\Omega}_{T_0}^\alpha = \{\omega \in \Omega,\ W(\cdot,\omega) \in C^\alpha([0,T_0])\}$ with $0 < \alpha < 1/2$. Note also that we could have replaced in Section 3 $W(\cdot,\omega)$ by any function $g(.)$ in $C^\alpha([0,T_0])$ with $\alpha > 0$ (all the constants appearing in the estimates would then depend on $|g|_{C^\alpha([0,T_0])}$, instead of ω and T_0). Hence, we may fix $\omega \in \tilde{\Omega}_{T_0}^\alpha$ and we regard $U^\omega(t,s)f$ as a function of the Brownian path $W(\cdot,\omega)$. We then denote $U^\omega(t,s)$ by $U^W(t,s)$ to clarify the dependence.

Proposition 8. *Let $T_0 > 0$, $R > 0$ and $M > 0$ be fixed. There exist a $T_R > 0$, and a constant $C_{R,T_0,M} > 0$ such that if $f \in C_0^\infty(\mathbb{R}^d)$, with supp$(f) \subset B(0,M)$, and if $W_1, W_2 \in B_R$, then, for any $t,s \in [0,T_0]$ with $|t-s| < T_R$ we have*

$$|(U^{W_1}(t,s) - U^{W_2}(t,s))f|_{L^2} \le C_{R,T_0,M}|W_1 - W_2|_{C([0,T_0])}\Big(|f|_{L^2} + \sum_{|\alpha|\le \frac{d}{2}+3} |\partial_y^\alpha f|_{L^1}\Big), \tag{5.1}$$

where B_R is the centered ball in $C^\alpha([0,T_0])$ with radius R, and $U^W(t,s)$ is the unique propagator of (1.3).

Proof. see Appendix. □

Before giving the proof of Proposition 2, we state a corollary of Proposition 8.

Corollary 5.1. *For T_0, T_R as in Proposition 8, $f \in L^2(\mathbb{R}^d)$, and any $s \in [0,T_0]$, $W \mapsto U^W(\cdot,s)f$ is continuous in the sense of Proposition 8, from $C([0,T_0]) \cap B_R$ into $C([s, s+T_R \wedge T_0]; L^2(\mathbb{R}^d))$.*

Proof. The proof of Corollary 5.1 follows easily from Proposition 8 and (3) of Proposition 6 (with $k=0$) which states that $f \mapsto U^W(\cdot,s)f$ is continuous on $L^2(\mathbb{R}^d)$, uniformly for $W \in B_R$. □

Proof of Proposition 2. We first consider a truncated version of equation (2.2). Let, for $M > 0$, χ_M be a positive $C_0^\infty(\mathbb{R}^+)$ function with supp$\chi_M \subset [0,2M]$, $\chi_M \equiv 1$ on $[0,M]$, and $0 \le \chi_M \le 1$. We set $f_M(u) = \chi_M(|u|^2)|u|^{2\sigma}u$, and consider the following equation, which clearly possesses a unique solution, denoted by $u^{W,M}$, in $C(\mathbb{R}^+; L^2(\mathbb{R}^d))$, since f_M is globally Lipschitz in $L^2(\mathbb{R}^d)$:

$$i\partial_t u = -\frac{1}{2}\big(\nabla - i(tN + W(t)\Gamma)\big)^2 u + \lambda f_M(u), \tag{5.2}$$

with $u(0,x) = u_0 \in L^2(\mathbb{R}^d)$. Equivalently, $u^{W,M}$ is the unique solution in $C(\mathbb{R}^+; L^2(\mathbb{R}^d))$ of the mild equation

$$u(t) = U^W(t,0)u_0 - i\lambda\int_0^t U^W(t,\tau)f_M(u(\tau))d\tau. \tag{5.3}$$

Let us prove that $W \mapsto u^{W,M}$ is continuous in the sense of Proposition 2 for any $M > 0$. It clearly follows from the estimates of Section 3 that $u^{W,M}$ is the limit in $C([0,T_0]; L^2)$ of the sequence $u_k^{W,M}$ defined by $u_0^{W,M}(t) = U^W(t,0)u_0$ and

$$u_{k+1}^{W,M} = U^W(t,0)u_0 - i\lambda \int_0^t U^W(t,\tau) f_M(u_k^{W,M}(\tau))d\tau, \tag{5.4}$$

and that this limit is uniform with respect to $W \in B_R$. Hence, in order to get the continuity of $W \mapsto u^{W,M}$, it is sufficient to prove that $u_k^{W,M}$ is continuous with respect to W in the sense of Proposition 2, for any k. We use an induction argument. Thanks to Corollary 5.1 and (5.4), this continuity will hold true at level $k+1$, assuming it at level k, if we prove that for any $W_0 \in B_R$,

$$\lim \sup_{t\in[0,T_0]} \int_0^t |U^W(t,s) f_M(u_k^{W,M}(s)) - U^{W_0}(t,s) f_M(u_k^{W_0,M}(s))|_{L^2} ds = 0, \tag{5.5}$$

when W tends to W_0 in $C([0,T_0])$, with $W \in B_R$. But the left hand side of (5.5) is bounded above, for $W \in B_R$, by

$$\begin{aligned}
& \int_0^{T_0} \sup_{t\in[0,T_0]} |U^W(t,s)(f_M(u_k^{W,M}(s)) - f_M(u_k^{W_0,M}(s)))|_{L^2} ds \\
& \quad + \int_0^{T_0} \sup_{t\in[0,T_0]} |U^W(t,s) f_M(u_k^{W_0,M}(s)) - U^{W_0}(t,s) f_M(u_k^{W_0,M}(s))|_{L^2} ds \\
\leq \; & C_{R,T_0,M} \int_0^{T_0} |u_k^{W,M}(s) - u_k^{W_0,M}(s)|_{L^2} ds \\
& \quad + \int_0^{T_0} \sup_{t\in[0,T_0]} |U^W(t,s) f_M(u_k^{W_0,M}(s)) - U^{W_0}(t,s) f_M(u_k^{W_0,M}(s))|_{L^2} ds.
\end{aligned}$$

The first term goes to zero by the induction assumption, and the second one by the dominated convergence Theorem, since Corollary 5.1 implies the convergence of the term inside the integral, while the boundedness of $U^W(t,s)$ in L^2, which is uniform with respect to $W \in B_R$, implies

$$|U^W(t,s) f_M(u_k^{W_0,M}(s)) - U^{W_0}(t,s) f_M(u_k^{W_0,M}(s))|_{L^2} \leq 2C_{R,T_0,M} |u_k^{W_0,M}|_{L^2}.$$

Hence, $u^{W,M}$ is continuous with respect to $W \in B_R$ in the sense of Proposition 2. It remains to get rid of the cut-off function χ_M. Note that we may restrict ourselves to a sufficiently small time interval $[0,T]$, provided that it depends only on R, T_0 and $|u_0|_{L^2}$. Let $(\rho, 2\sigma+2)$ and (r,p) be admissible pairs, with $p > 2\sigma+2$ and $r'(2\sigma+1) < \rho$ (this is possible, since $\sigma < 2/d$ implies $\rho'(2\sigma+1) < \rho$). Note that by Strichartz estimates (Proposition 7) applied to (5.3), $u^{W,M} \in L^\rho(0,T; L^{2\sigma+2})$, and for T sufficiently small, depending only on R, T_0 and $|u_0|_{L^2}$,

$$|u^{W,M}|_{L^\rho(0,T;L^{2\sigma+2})} \leq C_{R,T_0} |u_0|_{L^2}$$

and

$$|u^W|_{L^\rho(0,T;L^{2\sigma+2})} \leq C_{R,T_0} |u_0|_{L^2}.$$

Then, using again Proposition 7 for the difference between (5.3) and (2.3), we get for T as above,

$$\begin{aligned}
|u^{W,M} - u^W|_{L^\rho(0,T;L^{2\sigma+2}) \cap L^\infty(0,T;L^2)} \;\leq\; & C_{R,T_0} \left| \Big(\chi_M(|u^{W,M}|)^2 - 1\Big) |u^{W,M}|^{2\sigma} u^{W,M} \right|_{L^{r'}(0,T;L^{p'})} \\
& + C_{R,T_0,|u_0|_{L^2},\sigma} T^\gamma |u^{W,M} - u^W|_{L^\rho(0,T;L^{2\sigma+2})}
\end{aligned}$$

with $\gamma = 1 - \frac{2\sigma+2}{\rho}$, from which it follows, taking again T small enough depending on R, T_0, $|u_0|_{L^2}$, σ, that

$$|u^{W,M} - u^W|_{L^\rho(0,T;L^{2\sigma+2})\cap L^\infty(0,T;L^2)} \leq C_{R,T_0}\Big|\Big(\chi_M(|u^{W,M}|)^2 - 1\Big)|u^{W,M}|^{2\sigma}u^{W,M}\Big|_{L^{r'}(0,T;L^{p'})}. \tag{5.6}$$

On the other hand,

$$\begin{aligned}
&\Big|\Big(\chi_M(|u^{W,M}|)^2 - 1\Big)|u^{W,M}|^{2\sigma}u^{W,M}\Big|_{L^{r'}(0,T;L^{p'})} \\
\leq\ & \Big(\int_0^T |\mathbb{1}_{\{|u^{W,M}(s,\cdot)|^2\geq M\}}|u^{W,M}(s)|^{2\sigma+1}|_{L^{p'}}^{r'}ds\Big)^{1/r'} \\
\leq\ & \Big(\int_0^T |\mathbb{1}_{\{|u^{W,M}(s,\cdot)|^2\geq M\}}|_{L^q}^{r'}|u^{W,M}(s)|_{L^{2\sigma+2}}^{(2\sigma+1)r'}ds\Big)^{1/r'}
\end{aligned} \tag{5.7}$$

with $q > 1$ such that $\frac{1}{p'} = \frac{2\sigma+1}{2\sigma+2} + \frac{1}{q}$. Now, by Hölder inequality in time, this term is estimated by

$$\begin{aligned}
&\Big(\int_0^T |\mathbb{1}_{\{|u^{W,M}(s,\cdot)|^2\geq M\}}|_{L^q}^{\beta}ds\Big)^{1/\beta}|u^{W,M}|_{L^\rho(0,T;L^{2\sigma+2})}^{2\sigma+1} \\
\leq\ & (C_{R,T_0}|u_0|_{L^2})^{2\sigma+1}\Big(\int_0^T (\text{meas}\{|u^{W,M}(s,\cdot)|^2 \geq M\})^{\beta/q}ds\Big)^{1/\beta}
\end{aligned} \tag{5.8}$$

where $\frac{1}{r'} = \frac{2\sigma+1}{\rho} + \frac{1}{\beta}$; note that $\frac{1}{\beta} > 0$, since $(2\sigma+1)r' < \rho$. In turn, we have

$$\begin{aligned}
&\Big(\int_0^T (\text{meas}\{|u^{W,M}(s,\cdot)|^2 \geq M\})^{\beta/q}ds\Big)^{1/\beta} \\
\leq\ & \Big(\int_0^T (\frac{1}{M}\int_{\{|u^{W,M}(s,x)|^2\geq M\}}|u^{W,M}(s,x)|^2dx)^{\beta/q}ds\Big)^{1/\beta} \\
\leq\ & \frac{1}{M^{1/q}}\Big(\int_0^T |u^{W,M}(s,\cdot)|_{L^2}^{2\beta/q}ds\Big)^{1/\beta} = \frac{1}{M^{1/q}}T^{1/\beta}|u_0|_{L^2}^{2/q}.
\end{aligned} \tag{5.9}$$

We have used in the last equality the fact that $|u^{W,M}(s,\cdot)|_{L^2} = |u_0|_{L^2}$, for any s. This fact is easily seen, on a formal point of view, by multiplying equation (5.2) by $\bar{u}$, integrating on $\mathbb{R}^d$ and taking the imaginary part, and may be justified, as is classical, using a regularization procedure.

Finally, collecting (5.6), (5.7), (5.8) and (5.9) shows that $u^{W,M}$ converges to u^W in $L^\infty(0,T;L^2)$ as M goes to infinity, uniformly for $W \in B_R$, and Proposition 2 follows. □

Proof of Theorem 3. By Assumption (A), B_R being a Borel set of $C([0,T_0])$, we have

$$\lim_{\varepsilon\to 0}\mathbb{P}(M^\varepsilon \in B_R) = \mathbb{P}(W \in B_R),$$

where we put $M^\varepsilon(t) = \frac{\varepsilon}{\sigma_0}\int_0^{t/\varepsilon^2} m(s)ds$. On the other hand, since $\mathbb{P}\big(W \in C^\alpha([0,T_0])\big) = 1$ if $\alpha < 1/2$, for any $\eta > 0$ there exists $R_0 > 0$ such that

$$\mathbb{P}(W \in B_{R_0}) \geq 1 - \eta/2.$$

Thus, there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0,\varepsilon_0)$

$$\mathbb{P}(M^\varepsilon \in B_{R_0}) \geq 1 - \eta,$$

and hence

$$\mathbb{P}(W \in B_{R_0} \text{ and } M^\varepsilon \in B_{R_0}) \geq 1 - 3\eta/2.$$

By Skohorod Theorem, there exist a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, a subsequence of random variables $\{\tilde{M}^\varepsilon\}$ and a Brownian motion $\tilde{W}$ such that $\mathcal{L}(\tilde{M}^\varepsilon) = \mathcal{L}(M^\varepsilon)$, $\mathcal{L}(\tilde{W}) = \mathcal{L}(W)$ and $\tilde{M}^\varepsilon$ converges to $\tilde{W}$, $\tilde{\mathbb{P}}$-a.s., in $C([0, T_0])$. Then, for any $\varepsilon \in (0, \varepsilon_0)$,

$$\tilde{\mathbb{P}}\big(\tilde{M}^\varepsilon \in B_{R_0} \text{ and } \tilde{W} \in B_{R_0}\big) \geq 1 - 3\eta/2.$$

Now, if $\tilde{M}^\varepsilon \in B_{R_0}$, then the solution $u^{\tilde{M}^\varepsilon}$ of equation (2.3) with W replaced by $\tilde{M}^\varepsilon$ is well defined in $C([0, T_0]; L^2(\mathbb{R}^d))$ by the proof of (i) of Proposition 1, and by Proposition 2, for any $\delta > 0$,

$$\tilde{\mathbb{P}}\big(\tilde{M}^\varepsilon \in B_{R_0},\ \tilde{W} \in B_{R_0} \text{ and } |u^{\tilde{M}^\varepsilon} - u^{\tilde{W}}|_{C([0,T_0];L^2)}\big) > \delta)$$

converges to 0 as ε goes to 0. Therefore, for any $\delta > 0$,

$$\tilde{\mathbb{P}}(|u^{\tilde{M}^\varepsilon} - u^{\tilde{W}}|_{C([0,T];L^2)} > \delta) \leq 2\eta$$

provided that ε is sufficiently small. In particular, u^{M^ε} converges to u^W in distribution in $C([0, T_0]; L^2)$.

One easily prove in the same way that $(M^\varepsilon, u^{M^\varepsilon})$ converges in distribution in $C([0, T_0]) \times C([0, T_0]; L^2(\mathbb{R}^d))$ to (W, u^W), and we deduce that

$$\psi^{M^\varepsilon}(t, x) = e^{-i|x|^2(tN + M^\varepsilon(t)\Gamma)} u^{M^\varepsilon}(t, x),$$

which is clearly the unique solution in $C([0, T_0]; L^2(\mathbb{R}^d))$ of equation (2.5), converges in distribution in $C([0, T_0]; L^2)$ to the solution ψ of equation (1.5) given by Theorem 2. $\square$

Remark 5.1. *The arguments of the proof of Proposition 2 and Theorem 3 are still valid in the presence of a damping term, simply setting $f_M(u) = \chi_M(|u|^2)|u|^{2\sigma}u - i\gamma u$, $\gamma > 0$ and noting that L^2 norm is decreasing in the proof of Proposition 2. Thus the result of Theorem 3 is still true with a damping term.*

Remark 5.2. *In the case where $\sigma \geq 2/d$, using the estimates $|u^{W,M}|_{Y_T}, |u^W|_{Y_T} \leq C_{R,T_0}|u_0|_\Sigma$ in the proof of Proposition 2, as in the proof of local existence of solutions in Σ, it is possible to get*

$$|u^{W,M} - u^W|_{L^\rho_T L^{2\sigma+2} \cap L^\infty_T L^2} \leq \Big(C_{R,T_0}|u_0|_\Sigma\Big)^{2\sigma+1} \frac{1}{M^{1/q}} T^{1/r'} |u_0|_{L^2}^{2/q}.$$

for small time $T > 0$. For any fixed time $T_1 < T^*_{u_0,\omega}$, we obtain the same estimate for $|u^{W,M} - u^W|_{L^\rho_{T_1} L^{2\sigma+2} \cap L^\infty_{T_1} L^2}$ but with $|u_0|_\Sigma$ replaced by $\sup_{t\in[0,T_1]} |u(t)|_\Sigma$. It is not clear that $\sup_{t\in[0,T_1]} |u(t)|_\Sigma$ depends "only" on $W \in B_R$, therefore the convergence $u^{W,M} \to u^W$ as $M \to +\infty$ may not be uniform in $W \in B_R$.

6. Appendix

Proof of Proposition 5. Since we already know that $S(t, s, x, y)$ is C^1 for $0 < |t - s| \leq T_\omega$, it suffices to prove (3.15)-(3.17) and similar asymptotics for the partial derivatives in the space variable, which will prove that $\partial_x^\alpha \partial_y^\beta \tilde{S}(t, s, x, y)$ is C^1 for $|t - s| \leq T_\omega$. We recall that $0 < \alpha < 1/2$ is fixed, and that ω is such that $W(\cdot, \omega) \in C^\alpha([0, T_0])$. Using the system (3.1), with initial conditions $(x(s), \xi(s)) = (y, \eta)$, and the estimate (3.2), we get for any $\sigma \in [s,t]$ with $s,t \in [0,T_0]$:

$$\begin{aligned}
& |\xi(\sigma,s,y,\eta) - \eta - (\sigma-s)(sN+W(s)\Gamma)\eta + (\sigma-s)(sN+W(s)\Gamma)^2 y| \\
= \; & \Big| \int_s^\sigma (uN+W(u)\Gamma)(\xi(u) - uNx(u) - W(u)\Gamma x(u))du \\
& \quad - \int_s^\sigma (sN+W(s)\Gamma)\eta du + \int_s^\sigma (sN+W(s)\Gamma)^2 y du \Big| \\
= \; & \Big| \int_s^\sigma ((u-s)N + (W(u)-W(s))\Gamma)(\xi(u) - (uN+W(u)\Gamma)x(u))du \\
& \quad + \int_s^\sigma (sN+W(s)\Gamma)(\xi(u) - \eta - (uN+W(u)\Gamma)(x(u)-y))du \\
& \quad -(sN+W(s)\Gamma)\int_s^\sigma ((u-s)N + (W(u)-W(s))\Gamma) y du \Big| \\
& \leq C_{\omega,T_0}(1+|y|+|\eta|)|\sigma-s|^{1+\alpha},
\end{aligned} \tag{6.1}$$

where $\xi(u) = \xi(u,s,y,\eta)$ and $x(u) = x(u,s,y,\eta)$. Similarly,

$$|x(\sigma,s,y,\eta) - y - (\sigma-s)\eta + (\sigma-s)(sN+W(s)\Gamma)y| \leq C_{\omega,T_0}(1+|y|+|\eta|)|\sigma-s|^{1+\alpha}. \tag{6.2}$$

Thus, we get

$$\begin{aligned}
& \Big| x(t,s,y,\eta) - y - (t-s)\eta + \int_s^t (\sigma N + W(\sigma)\Gamma) y d\sigma \Big| \\
= \; & \Big| \int_s^t [\xi(\sigma,s,y,\eta) - \eta - (\sigma-s)(sN+W(s)\Gamma)\eta + (\sigma-s)(sN+W(s)\Gamma)^2 y \\
& \quad +((s-\sigma)N + (W(s)-W(\sigma))\Gamma)(x(\sigma,s,y,\eta)-y) \\
& \quad -(sN+W(s)\Gamma)(x(\sigma,s,y,\eta) - y - (\sigma-s)\eta + (\sigma-s)(sN+W(s)\Gamma)y)]d\sigma \Big| \\
\leq \; & C_{\omega,T_0}|t-s|^{2+\alpha}(1+|y|+|\eta|).
\end{aligned}$$

We deduce, using (3.3), that

$$|\tilde{x}(t,s,y,\eta) - y - \eta + \int_s^t (\sigma N + W(\sigma)\Gamma) y d\sigma| \leq C_{\omega,T_0}|t-s|^{1+\alpha}(1+|y|+|\eta|),$$

and also that

$$|\tilde{x}(t,s,y,\eta) - y - \eta + (t-s)(sN+W(s)\Gamma)y| \leq C_{\omega,T_0}|t-s|^{1+\alpha}(1+|y|+|\eta|). \tag{6.3}$$

Hence, plugging $\eta = \tilde{\eta}(t,s,y,x)$ into (6.3) and using the fact that

$$|\tilde{\eta}(t,s,y,x)| \leq C_{\omega,T_0}(1+|x|+|y|),$$

as follows from Proposition 3, we deduce

$$\begin{aligned}
& |\tilde{\eta}(t,s,y,x) - x + y - (t-s)(sN+W(s)\Gamma)y| \\
= \; & |\tilde{\eta}(t,s,y,x) - \tilde{x}(t,s,y,\tilde{\eta}(t,s,y,x)) + y - (t-s)(sN+W(s)\Gamma)y| \\
\leq \; & C_{\omega,T_0}|t-s|^{1+\alpha}(1+|x|+|y|)
\end{aligned} \tag{6.4}$$

and

$$\Big| \eta(t,s,y,x) - \frac{x-y}{t-s} - (sN+W(s)\Gamma)y \Big| \leq C_{\omega,T_0}|t-s|^{\alpha}(1+|x|+|y|). \tag{6.5}$$

From (6.2) and (6.5), we get for $s \leq \tau \leq t$:

$$\begin{aligned} & |x(\tau, s, y, \eta(t, s, y, x)) - y - (\tau - s)\eta(t, s, y, x) + (\tau - s)(sN + W(s)\Gamma)y| \\ \leq \;& C_{\omega,T_0}(1 + |y| + |\eta(t, s, y, x)|)|\tau - s|^{1+\alpha} \\ \leq \;& C_{\omega,T_0}(1 + |x| + |y|)|\tau - s|^{\alpha}. \end{aligned}$$

Hence, using again (6.5) :

$$\left|x(\tau, s, y, \eta(t, s, y, x) - y - \frac{\tau - s}{t - s}(x - y)\right| \leq C_{\omega,T_0}|\tau - s|^{\alpha}(1 + |x| + |y|). \tag{6.6}$$

On the other hand, (6.1) and (6.5) imply for $s \leq \tau \leq t$:

$$\begin{aligned} & \left|\xi(\tau, s, y, \eta(t, s, y, x)) - \frac{x - y}{t - s} - (sN + W(s)\Gamma)y - \frac{\tau - s}{t - s}(sN + W(s)\Gamma)(x - y)\right| \\ \leq \;& C_{\omega,T_0}|\tau - s|^{\alpha}(1 + |x| + |y|). \end{aligned} \tag{6.7}$$

Now, from (3.7)

$$L(\tau, q(\tau), v(\tau)) = \frac{1}{2}\xi^2(\tau, s, y, \eta(t, s, y, x)) - \frac{1}{2}[(\tau N + W(\tau)\Gamma)x(\tau, s, y, \eta(t, s, y, x))]^2.$$

Hence,we get from (6.6) and (6.7)

$$\begin{aligned} & \left|L(\tau, q(\tau), v(\tau)) - \frac{1}{2}\left(\frac{x - y}{t - s}\right)^2 - \frac{x - y}{t - s} \cdot (sN + W(s)\Gamma)y - \frac{\tau - s}{(t - s)^2}(x - y) \cdot (sN + W(s)\Gamma)(x - y)\right| \\ \leq \;& C_{\omega,T_0}|\tau - s|^{\alpha-1}(1 + |x|^2 + |y|^2) \end{aligned}$$

from which we deduce that

$$\tilde{S}(t, s, x, y) = (t - s)\int_s^t L(\tau, q(\tau), v(\tau))d\tau$$

satisfies

$$|\tilde{S}(t, s, x, y) - \frac{1}{2}(x - y)^2 - \frac{1}{2}(t - s)(x - y) \cdot (sN + W(s)\Gamma)(x + y)| \leq C_{\omega,T_0}|t - s|^{1+\alpha}(1 + |x|^2 + |y|^2)$$

which is (3.15) except for the space derivatives.

We now consider the space derivatives of $\tilde{S}$. Note that (6.7) implies

$$|\tilde{\xi}(t, s, y, \tilde{\eta}(t, s, y, x)) - (x - y) - (t - s)(sN + W(s)\Gamma)x| \leq C_{\omega,T_0}|t - s|^{1+\alpha}(1 + |x| + |y|) \tag{6.8}$$

which, together with (3.13), gives (3.15) for $\partial_x\tilde{S}$. The estimate for $\partial_y\tilde{S}$ follows from (3.14) and (6.4). Next, we note that $(y, \eta) \mapsto (y, \tilde{x}(t, s, y, \eta))$ and $(y, \eta) \mapsto (y, \tilde{\xi}(t, s, y, \eta))$ are linear, hence the same is true for $(y, x) \mapsto (y, \tilde{\eta}(t, s, y, x))$ and $(y, x) \mapsto (y, \tilde{\xi}(t, s, y, \tilde{\eta}(t, s, y, x)))$. It follows that

$$\begin{aligned} \partial_{x_j}\tilde{S}(t, s, x, y) &= \tilde{\xi}_j(t, s, y, \tilde{\eta}(t, s, y, x)) \\ &= \sum_{k=1}^{d} y_k\tilde{\xi}_j(t, s, e_k, \tilde{\eta}(t, s, e_k, 0)) + \sum_{k=1}^{d} x_k\tilde{\xi}_j(t, s, 0, \tilde{\eta}(t, s, 0, e_k)). \end{aligned} \tag{6.9}$$

Hence by (6.8)

$$\begin{aligned} \partial_{x_l}\partial_{x_j}\tilde{S}(t, s, y, x) &= \tilde{\xi}_j(t, s, 0, \tilde{\eta}(t, s, 0, e_l)) \\ &= \delta_{jl} + (t - s)(s\nu_j + W(s)\kappa_j)\delta_{jl} + O(|t - s|^{1+\alpha}) \end{aligned}$$

and

$$\begin{aligned}\partial_{y_l}\partial_{x_j}\tilde{S}(t,s,y,x) &= \tilde{\xi}_j(t,s,e_l,\tilde{\eta}(t,s,e_l,0)) \\ &= -\delta_{jl}+O(|t-s|^{1+\alpha}).\end{aligned}$$

Estimates on $\partial_{y_l}\partial_{y_j}\tilde{S}$ follow in the same way, using (3.14) and (6.4). It is clear that $\partial_x^\alpha\partial_y^\beta S(t,s,x,y)=0$ for $|\alpha+\beta|\geq 3$. In order to prove (3.16) and (3.17), we make use of Hamilton-Jacobi equation (3.10)-(3.11). We only consider the derivative with respect to t, which, thanks to (3.15) and (6.8), may be written as

$$\begin{aligned}\partial_t\tilde{S}(t,s,x,y) &= S(t,s,x,y)+(t-s)(\partial_t S)(t,s,x,y) \\ &= S(t,s,x,y)-\frac{1}{2(t-s)}\Big[(\nabla_x\tilde{S})(t,s,x,y)-(t-s)(tN+W(t)\Gamma)x\Big]^2 \\ &= \frac{1}{t-s}\Big\{\tilde{S}(t,s,x,y)-\frac{1}{2}\Big[\tilde{\xi}(t,s,y,\tilde{\eta}(t,s,y,x))-(t-s)(tN+W(t)\Gamma)x\Big]^2\Big\} \\ &= \frac{1}{2}(x-y)\cdot(sN+W(s)\Gamma)(x+y)+(1+|x|^2+|y|^2)O(|t-s|^\alpha).\end{aligned}$$

Estimate (3.17) is obtained in the same way with the use of (3.15) and (6.4), and the space derivatives are estimated as above. □

Proof of Proposition 6. Note that the operator $I(t,s,a)$ is bounded in $L^2(\mathbb{R}^d)$ by Asada-Fujiwara's Theorem (see [4]). Indeed, by (3.14) and (6.4),

$$(\partial_{y_k}\partial_{y_l}\tilde{S})(t,s,x,y)=-\tilde{\eta}_l(t,s,e_k,0)=\delta_{kl}+O(|t-s|), \tag{6.10}$$

and therefore,

$$|\det(\partial_{y_k}\partial_{y_l}\tilde{S})|\geq\frac{1}{2}\quad\text{for}\quad |t-s|\leq T_\omega \tag{6.11}$$

if T_ω is sufficiently small. Then, (1) is a direct consequence of the fact that $S(s,t,y,x)=-S(t,s,x,y)$, which itself follows from (3.6).

(2) is a special case of Proposition 3.2 in [28], but we repeat the proof for the sake of completeness. Let $f\in C_0^\infty(\mathbb{R}^d)$, then integrating by parts yields

$$[-i(t-s)\partial_{x_j}I(t,s,a)f-(t-s)I(t,s,a)(-i\partial_{x_j}f)](x)=I(t,s,(\partial_{x_j}\tilde{S}+\partial_{y_j}\tilde{S})a)f(x). \tag{6.12}$$

On the other hand, by (3.13) and (3.14), using the linearity of $(y,\eta)\mapsto\tilde{\xi}(t,s,y,\eta)$ and $(y,\eta)\mapsto\tilde{\eta}(t,s,y,x)$, we may write

$$\partial_{x_j}\tilde{S}(t,s,x,y)+\partial_{y_j}\tilde{S}(t,s,x,y)=\sum_{k=1}^d y_k\tilde{\xi}_{jk,1}(t,s)+\sum_{k=1}^d\tilde{\eta}_k(t,s,y,x)\tilde{\xi}_{jk,2}(t,s)$$

where we have set

$$\tilde{\xi}_{jk,1}(t,s)=\tilde{\xi}_j(t,s,e_k,0),\quad\text{and}\quad\tilde{\xi}_{jk,2}(t,s)=\tilde{\xi}_j(t,s,0,e_k)-\delta_{jk}.$$

This proves

$$I(t,s,(\partial_{x_j}\tilde{S}+\partial_{y_j}\tilde{S})a)=\sum_{k=1}^d I(t,s,\tilde{\xi}_{jk,1}a)x_k+\sum_{k=1}^d I(t,s,\tilde{\xi}_{jk,2}\tilde{\eta}_k a). \tag{6.13}$$

Now, using again the expression $\tilde{\eta}_k(t,s,y,x)=-\partial_{y_k}\tilde{S}(t,s,x,y)$ and integrating by parts yields

$$I(t,s,\tilde{\xi}_{jk,2}\tilde{\eta}_k\ a)f(x)=(t-s)I(t,s,\tilde{\xi}_{jk,2}\ a)(-i\partial_{x_k}f)(x). \tag{6.14}$$

Gathering (6.12), (6.13) and (6.14), and setting

$$a_{jk,21}=-\frac{\tilde{\xi}_{jk,1}}{t-s}a,\quad\text{and}\quad a_{jk,22}=\frac{\tilde{\xi}_{jk,2}}{t-s}a$$

leads to (3.23). Note that $a_{jk,21}$ and $a_{jk,22}$ are bounded for $t, s \in [0, T_0]$ with $|t-s| \le T_\omega$, as follows from the inequality

$$|\tilde{\xi}(t,s,y,\eta) - \eta| \le C_{\omega,T_0}|t-s|(1+|y|+|\eta|).$$

This inequality is easily verified by substituting (3.3) into (6.1). In order to prove (3.22), we use the fact that

$$\tilde{x}_j(t,s,y,\tilde{\eta}(t,s,y,x)) = x_j = \sum_{k=1}^{d} y_k \tilde{x}_{jk,1}(t,s) + \sum_{k=1}^{d} \tilde{\eta}_k(t,s,y,x)\tilde{x}_{jk,2}(t,s)$$

where we have set

$$\tilde{x}_{jk,1}(t,s) = \tilde{x}_j(t,s,e_k,0), \quad \text{and} \quad \tilde{x}_{jk,2}(t,s) = \tilde{x}_j(t,s,0,e_k),$$

which gives

$$x_j I(t,s,a) = \sum_{k=1}^{d} I(t,s,\tilde{x}_{jk,1}a)x_k + \sum_{k=1}^{d} I(t,s,\tilde{x}_{jk,2}\tilde{\eta}_k\ a);$$

the same computations as above then show that

$$I(t,s,\tilde{x}_{jk,2}\tilde{\eta}_k a) = (t-s)I(t,s,\tilde{x}_{jk,2}a)(-i\partial_{x_k})$$

and (3.22) follows after setting

$$a_{jk,11} = \frac{1}{t-s}(\tilde{x}_{jk,1} - \delta_{jk})a, \quad \text{and} \quad a_{jk,12} = -(\tilde{x}_{jk,2} - \delta_{jk})a.$$

Again, those functions are bounded since (6.3) implies

$$|\tilde{x}(t,s,y,\eta) - y - \eta| \le C_{\omega,T_0}|t-s|(1+|y|+|\eta|). \tag{6.15}$$

The combination of (2) and Asada-Fujiwara's Theorem implies (3), that is, the boundedness of $I(t,s,a)$ in $\Sigma(k)$ for any $k \in \mathbb{N}$.

The proof of (4) is essentially due to Fujiwara [13] and Yajima [28]. However, we repeat the arguments here for the sake of completeness. Let $f \in C_0^\infty(\mathbb{R}^d)$ and let $M > 0$ be such that supp$f \subset \{x \in \mathbb{R}^d, |x| \le M\}$. We recall that $W(\cdot,\omega) \in C^\alpha([0,T_0])$, and that using estimate (3.12) we may prove that if

$$C_M = C_{M,\omega,T_0} := \max\{M, \sup_{\substack{s,t\in[0,T_0] \\ |y|\le M}} |\nabla_y \tilde{S}(t,s,0,y)|\},$$

then

$$|\nabla_y \tilde{S}(t,s,x,y)| \ge \frac{|x|}{8} \quad \text{for} \quad |x| \ge 8C_M, \quad y \in \text{supp} f, \tag{6.16}$$

provided that $|t-s| \le T_\omega$, for some sufficiently small T_ω. Again, we use the fact that if $\nu = t-s$,

$$\frac{\nu}{i}\Big(\frac{\nabla_y \tilde{S}}{|\nabla_y \tilde{S}|^2} \cdot \nabla_y\Big) e^{i\tilde{S}/\nu} = e^{i\tilde{S}/\nu},$$

and integrate L times by parts to get, for $|x| \ge 8C_M$,

$$\begin{aligned}
|U^\omega(t,s)f(x)| &= \Big|(2\pi i\nu)^{-d/2}(\nu/i)^L \int_{\mathbb{R}^d} \Big[\Big(\frac{\nabla_y \tilde{S}}{(\nabla_y \tilde{S})^2} \cdot \nabla_y\Big)^L e^{i\tilde{S}/\nu}\Big] a(t,s) f(y) dy\Big| \\
&= \Big|(2\pi i\nu)^{-d/2}(\nu/i)^L \int_{\mathbb{R}^d} e^{i\tilde{S}/\nu} a(t,s) \Big\{\Big(\frac{\nabla_y \tilde{S}}{(\nabla_y \tilde{S})^2} \cdot \nabla_y\Big)^*\Big\}^L f(y) dy\Big| \\
&\le C_{\omega,T_0,M,L}\ \nu^{L-d/2}(1+|x|)^{-L} \int_{\mathbb{R}^d} \sum_{|\alpha|\le L} |\partial_y^\alpha f(y)| dy,
\end{aligned}$$

where we have used the bound (3.20) and the fact that

$$\Big|\Big[\Big(\frac{\nabla_y \tilde{S}}{(\nabla_y \tilde{S})^2}\cdot \nabla_y\Big)^*\Big]^L f(y)\Big| \le C_L(1+|x|)^{-L}\sum_{|\alpha|\le L}|\partial_y^\alpha f(y)|$$

by the estimate (6.16). Thus, choosing $L > d/2$

$$|U^\omega(t,s)f|_{L^2(|x|>8C_M)} \le C_{\omega,T_0,M,L}\ \nu^{L-d/2}\sum_{|\alpha|\le L}|\partial_y^\alpha f(y)|_{L^1(\mathbb{R}^d)}\Big(\int_{\mathbb{R}^d}(1+|x|)^{-2L}dx\Big)^{1/2},$$

and it follows that $|U^\omega(t,s)f|_{L^2(|x|>8C_M)}$ tends to zero as t goes to s.

For the case $|x| \le 8C_M$, we use the stationary phase method (see, e.g. Lemma A.8 of [14], or [15]). The equation $\nabla_y \tilde{S}(t,s,x,y) = 0$, as an equation in y, has a unique solution $y = \tilde{y}(t,s,x,0)$. Here, $(\tilde{y}(t,s,x,\eta),\eta)$ is the inverse map of $(y,\eta)\mapsto(\tilde{x}(t,s,y,\eta),\eta)$. We recall (6.11) and apply the stationary phase method; we obtain, for some smooth function $r(t,s,.)$,

$$U^\omega(t,s)f(x) = \Big|\det(\partial_{y_k}\partial_{y_l}\tilde{S})\Big|_{y=\tilde{y}}^{-1/2} e^{i\tilde{S}(t,s,x,\tilde{y})/\nu}(a(t,s)f(\tilde{y})+\nu r(t,s,\nu^{-1}x)) \tag{6.17}$$

where $\tilde{y} = \tilde{y}(t,s,x,0)$ and, for any $k\in\mathbb{N}$, there exist $K(k)\in\mathbb{N}$ and $C(k)>0$ such that for any $|\alpha|\le k$ and $t,s\in[0,T_0]$ with $|t-s|\le T_\omega$,

$$|\partial_x^\alpha r(t,s,\nu^{-1}x)| \le C_k|a(t,s)|\max_{|\beta|\le K(k)}\sup_{y\in\mathbb{R}^d}|\partial_y^\beta f(y)|.$$

On the other hand, it follows from (6.15), using $x = \tilde{x}(t,s,\tilde{y}(t,s,x,\eta))$, that

$$|\tilde{y}(t,s,x,\eta) - x + \eta| \le C_{\omega,T_0}|t-s|(1+|x|+|\eta|)$$

and in particular,

$$|\tilde{y}(t,s,x,0) - x| \le C_{\omega,T_0}|t-s|(1+|x|), \tag{6.18}$$

which implies that $\tilde{y}(t,s,x,0)$ converges to x as t goes to s, uniformly for $|x|\le 8C_M$. Thus,

$$\lim_{t\to s} f(\tilde{y}(t,s,x,0)) = f(x),$$

uniformly for $|x|\le 8C_M$. Also $\lim_{t\to s} a(t,s) = a(s,s) = 1$ and $\lim_{t\to s}|\det(\partial_{y_k}\partial_{y_l}\tilde{S})|_{y=\tilde{y}} = 1$ by (6.10). Finally, it follows from (3.15) and (6.18) that

$$\left|\frac{\tilde{S}(t,s,x,\tilde{y})}{\nu}\right| \le C_{\omega,T_0}|t-s|^\alpha(1+|x|^2),$$

hence $\tilde{S}/\nu$ tends to zero uniformly on $|x|\le 8C_M$. Accordingly, we get that $\lim_{t\to s} U^\omega(t,s)f(x) = f(x)$, uniformly for $|x|\le 8C_M$, which completes the proof of (4) of Proposition 6. □

Proof of Proposition 8. Let $W_1, W_2 \in B_R$ and $f\in C_0^\infty(\mathbb{R}^d)$. We write the difference as follows.

$$\begin{aligned}
&U^{W_1}(t,s)f(x) - U^{W_2}(t,s)f(x)\\
&= (2\pi i(t-s))^{-d/2}\int_{\mathbb{R}^d} e^{iS^{W_1}(t,s,x,y)}(a^{W_1}(t,s)-a^{W_2}(t,s))f(y)dy\\
&+(2\pi i(t-s))^{-d/2}\int_{\mathbb{R}^d} e^{iS^{W_1}(t,s,x,y)}(1-e^{iS^{W_2}(t,s,x,y)-iS^{W_1}(t,s,x,y)})(a^{W_2}(t,s))f(y)dy \qquad (6.19)\\
&= \mathrm{I}+\mathrm{II}
\end{aligned}$$

where a^{W_j} is defined as in (3.19), replacing W by W_j in all Section 3. We use the following lemma to estimate (6.19).

Lemma 6.1. *The difference $S^{W_1} - S^{W_2}$ satisfies the estimate :*

$$\sum_{|\alpha+\beta|\le 2} \left|\partial_x^\alpha \partial_y^\beta \Big(S^{W_1}(t,s,x,y) - S^{W_2}(t,s,x,y) \Big)\right| \le C_{R,T_0} |W_1 - W_2|_{C([0,T_0])} (1+|x|+|y|)^{2-|\alpha+\beta|}. \quad (6.20)$$

We postpone the proof of Lemma 6.1 to the end of the section and continue the proof of Proposition 8.

Part I of (6.19) is estimated using Asada-Fujiwara's Theorem in [4]; we refer to the beginning of the proof of Proposition 6 in this section for a justification of the fact that the assumptions of [4] are satisfied by the phase function $S^{W_1}(t,s,x,y)$ provided that $|t-s| \le T_R$, where T_R is sufficiently small. Then,

$$|\mathrm{I}|_{L^2} = |I^{W_1}(t,s,a^{W_1} - a^{W_2})|_{L^2} \le C_{R,T_0} |a^{W_1}(t,s) - a^{W_2}(t,s)| |f|_{L^2},$$

where the constant C_{R,T_0} depends only on R and T_0, and provided $|t-s| \le T_R$. Using the definition (3.19) of $a(t,s)$, we may write the difference as follows.

$$a^{W_1}(t,s) - a^{W_2}(t,s) = \exp\Big(\frac{1}{2}\int_s^t R^{W_1}(\tau,s)d\tau\Big)\Big[1 - \exp\Big(\frac{1}{2}\int_s^t (R^{W_2}(\tau,s) - R^{W_1}(\tau,s))d\tau\Big)\Big].$$

By (3.20), we have

$$\Big|\exp\Big(\frac{1}{2}\int_s^t R^{W_1}(\tau,s)d\tau\Big)\Big| = |a^{W_1}(t,s)| \le 1 + C_{R,T_0}|t-s|$$

and, by (3.18) and (6.20),

$$\begin{aligned} |R^{W_2}(\tau,s) - R^{W_1}(\tau,s)| &\le |\Delta_x S^{W_1}(\tau,s) - \Delta_x S^{W_2}(\tau,s)| + |\mathrm{Tr}(\Gamma)||W_2(\tau) - W_1(\tau)| \\ &\le C_{R,T_0}|W_1 - W_2|_{C([0,T_0])}. \end{aligned}$$

Therefore, we have

$$|a^{W_1}(t,s) - a^{W_2}(t,s)| \le C_{R,T_0}|t-s||W_1 - W_2|_{C([0,T_0])},$$

and we obtain, for $|t-s| \le T_R$,

$$|\mathrm{I}|_{L^2} \le C_{R,T_0}|t-s||W_1 - W_2|_{C([0,T_0])}|f|_{L^2}$$

for $f \in L^2(\mathbb{R}^d)$.

Next, we estimate II. Recall that $\mathrm{supp} f \subset \{y \in \mathbb{R}^d, |y| \le M\}$. Let $M' > 0$ be a constant that will be chosen later. We write II using the cut-off functions $\Theta_M, \Theta_{M'} \in C_0^\infty(\mathbb{R}^d)$ as follows.

$$\begin{aligned} \mathrm{II} &= (2\pi i\nu)^{-d/2}\int_{\mathbb{R}^d} e^{iS^{W_1}(t,s,x,y)}\Big(1 - e^{i(S^{W_2}(t,s,x,y) - S^{W_1}(t,s,x,y))}\Big) \\ &\qquad\qquad \times a^{W_2}(t,s)\Theta'_M(x)\Theta_M(y)f(y)dy \\ &\quad + (2\pi i\nu)^{-d/2}\int_{\mathbb{R}^d}\Big(e^{i\tilde{S}^{W_1}(t,s,x,y)/\nu} - e^{i\tilde{S}^{W_2}(t,s,x,y)/\nu}\Big) \\ &\qquad\qquad \times a^{W_2}(t,s)(1 - \Theta'_M(x))\Theta_M(y)f(y)dy, \\ &= \mathrm{III} + \mathrm{IV}, \end{aligned}$$

where we put $\nu = t-s$. Since all the space derivatives of the amplitude function

$$(1 - e^{i(S^{W_2}(t,s,x,y) - S^{W_1}(t,s,x,y))})a^{W_2}(t,s)\Theta'_M(x)\Theta_M(y)$$

in III are bounded, taking $S^{W_1}(t,s,x,y)$ for the phase function, and applying Asada-Fujiwara's Theorem [4], combined with Lemma 6.1, we get

$$|\mathrm{III}|_{L^2} \le C_{R,T_0,M,M'}|a^{W_2}(t,s)||W_1 - W_2|_{C([0,T_0])}|f|_{L^2}. \quad (6.21)$$

Concerning IV, we set

$$L_j = \Big(\frac{\nabla_y \tilde{S}^{W_j}}{|\nabla_y \tilde{S}^{W_j}|^2} \cdot \nabla_y\Big), \quad j = 1, 2,$$

and observe that $\frac{\nu}{i} L_j e^{i\tilde{S}^{W^j}/\nu} = e^{i\tilde{S}^{W^j}/\nu}$. We then integrate L times by parts to get

$$\begin{aligned}
\mathrm{IV} &= (2\pi i\nu)^{-d/2}(\nu/i)^L \int_{\mathbb{R}^d} \Big([L_1^L e^{i\tilde{S}^{W_1}(t,s,x,y)/\nu}] - [L_2^L e^{i\tilde{S}^{W_2}(t,s,x,y)/\nu}]\Big) \\
&\qquad\qquad \times a^{W_2}(t,s)(1-\Theta'_M(x))\Theta_M(y) f(y) dy \\
&= (2\pi i\nu)^{-d/2}(\nu/i)^L \int_{\mathbb{R}^d} e^{i\tilde{S}^{W_1}(t,s,x,y)/\nu} a^{W_2}(t,s)(1-\Theta'_M(x)) \\
&\qquad\qquad \times [(L_1^*)^L - (L_2^*)^L]\Theta_M(y) f(y) dy \\
&\quad + (2\pi i\nu)^{-d/2}(\nu/i)^L \int_{\mathbb{R}^d} (e^{i\tilde{S}^{W_1}(t,s,x,y)/\nu} - e^{i\tilde{S}^{W_2}(t,s,x,y)/\nu}) a^{W_2}(t,s)(1-\Theta'_M(x)) \\
&\qquad\qquad \times (L_2^*)^L \Theta_M(y) f(y) dy.
\end{aligned}$$

On the other hand, using estimate (3.12), we obtain as in [28],

$$\begin{aligned}
|\partial_y \tilde{S}^{W_j}(t,s,x,y)| &= \Big|\partial_y \tilde{S}^{W_j}(t,s,0,y) - \int_0^1 \frac{\partial^2 \tilde{S}^{W_j}}{\partial_x \partial_y}(t,s,\theta x, y)\cdot x d\theta\Big| \\
&\geq \frac{1}{4}|x| - |\partial_y \tilde{S}^{W_j}(t,s,0,y)| \geq \frac{1}{4}|x| - C_{M,R,T_0},
\end{aligned}$$

where $C_{M,R,T_0} = \max\{M, \sup_{s,t\in[0,T_0],|y|\leq M}(\partial_y \tilde{S}^{W_j})(t,s,0,y)|\}$, provided that $|t-s| \leq T_R$, for some sufficiently small T_R. Thus,

$$|\partial_y \tilde{S}^{W_j}(t,s,x,y)| \geq \frac{|x|}{8}, \quad \text{for} \quad |x| \geq 8C_{M,R,T_0}, \quad y \in \mathrm{supp} f. \tag{6.22}$$

We then choose $M' = 8C_{M,R,T_0}$ and get

$$|(1-\Theta'_M(x))(L_j^*)^L \Theta_M(y) f(y)| \leq C_{R,T_0,M}(1+|x|)^{-L} \sum_{|\alpha|\leq L} |\partial_y^\alpha f(y)|, \quad j = 1, 2. \tag{6.23}$$

Moreover, it may be checked that whenever $|x| \geq M'$,

$$\sum_{|\alpha|\leq K} |\partial_y^\alpha (L_1^* - L_2^*)\Theta_M(y) f(y)| \leq C_{R,T_0,M}(1+|x|)^{-1}|W_1 - W_2|_{C([0,T_0])} \sum_{|\beta|\leq K+1} |\partial_y^\beta f(y)|,$$

and

$$\sum_{|\beta|\leq K} |\partial_y^\beta (L_2^*)^k \Theta_M(y) f(y)| \leq C_{R,T_0,M}(1+|x|)^{-k} \sum_{|\gamma|\leq k+K} |\partial_y^\beta f(y)|.$$

Accordingly, for any $L \in \mathbb{N}$,

$$\begin{aligned}
&|(1-\Theta'_M(x))[(L_1^*)^L - (L_2^*)^L]\Theta_M(y) f(y)| \\
=\; & \Big|(1-\Theta'_M(x)) \sum_{k=0}^{L-1} (L_1^*)^{L-k-1}(L_1^* - L_2^*)(L_2^*)^k \Theta_M(y) f(y)\Big| \\
\leq\; & C_{R,T_0,M}(1+|x|)^{-L}|W_1 - W_2|_{C([0,T_0])} \sum_{|\alpha|\leq L} |\partial_y^\alpha f(y)|.
\end{aligned} \tag{6.24}$$

We apply (6.24) to the first term of IV, and apply (6.23) and Lemma 6.1 for the second term of IV. Then we obtain

$$\begin{aligned}\mathrm{IV} \quad &\le \quad C_{R,T_0,M}\ \nu^{L-d/2}(1+|x|)^{-L}|W_1-W_2|_{C([0,T_0])}|a^{W_2}(t,s)|\sum_{|\alpha|\le L}\Big|\partial_y^\alpha f(y)\Big|_{L^1}\\ &\quad +C_{R,T_0,M}\ \nu^{L-d/2}(1+|x|)^{-L+2}|W_1-W_2|_{C([0,T_0])}|a^{W_2}(t,s)|\sum_{|\alpha|\le L}\Big|\partial_y^\alpha f(y)\Big|_{L^1}.\end{aligned}$$

Hence, choosing $L=\frac{d}{2}+3$,

$$\begin{aligned}|\mathrm{IV}|_{L^2} \quad &\le \quad C_{R,T_0,M}\nu^{L-d/2}|a^{W_2}(t,s)||W_1-W_2|_{C([0,T_0])}\sum_{|\alpha|\le d/2+3}\Big|\partial_y^\alpha f(y))\Big|_{L^1}\\ &\qquad \times\Big\{\Big(\int_{\mathbb{R}^d}(1+|x|)^{-d-2}dx\Big)^{1/2}+\Big(\int_{\mathbb{R}^d}(1+|x|)^{-d-6}dx\Big)^{1/2}\Big\},\end{aligned}$$

which ends the proof of Proposition 8. □

We finally show Lemma 6.1 to complete Proposition 8.

Proof of Lemma 6.1. We denote by $(x^j(t,s,y^j,\eta^j),\xi^j(t,s,y^j,\eta^j))$ the solution of the system (3.1) with $W(t)$ replaced by $W_j(t)$ for $j=1,2$. Then, we have for $y^j,\eta^j\in\mathbb{R}^d$:

$$\begin{aligned}|\xi^1(t,s,y^1,\eta^1)-\eta^1-(\xi^2(t,s,y^2,\eta^2)-\eta^2)| &\le C_{R,T_0}|t-s||W_1-W_2|_{C([0,T_0])}\\ &\quad +C_{R,T_0}\int_s^t(|\xi^1(\tau)-\xi^2(\tau)|+|x^1(\tau)-x^2(\tau)|)d\tau\\ |x^1(t,s,y^1,\eta^1)-y^1-(x^2(t,s,y^2,\eta^2)-y^2)| &\le C_{R,T_0}|t-s||W_1-W_2|_{C([0,T_0])}\\ &\quad +C_{R,T_0}\int_s^t(|\xi^1(\tau)-\xi^2(\tau)|+|x^1(\tau)-x^2(\tau)|)d\tau,\end{aligned}$$

where $x^j(t)=x^j(t,s,y^j,\eta^j)$ and $\xi^j(t)=\xi^j(t,s,y^j,\eta^j)$ for $j=1,2$. Therefore, Gronwall lemma implies,

$$|\xi^1(t)-\xi^2(t)|+|x^1(t)-x^2(t)|\le C_{R,T_0}\left[|t-s||W_1-W_2|_{C([0,T_0])}+|\eta^1-\eta^2|+|y^1-y^2|\right]. \tag{6.25}$$

Using again the equation of (x^j,ξ^j), we estimate $(\tilde{x}^j,\tilde{\xi}^j)$ defined by $\tilde{x}^j(t,s,y^j,\eta^j)=x^j(t,s,y^j,\eta^j/(t-s))$ and $\tilde{\xi}^j(t,s,y^j,\eta^j)=(t-s)\xi^j(t,s,y^j,\eta^j/(t-s))$. We recall that $\frac{\sigma_0}{2}\nabla K(x)=\Gamma x$ and $\frac{1}{2}\nabla V(x)=Nx$. Hence,

$$\begin{aligned}&x^1(t,s,y^1,\eta^1)-x^2(t,s,y^2,\eta^2)\\ =\quad &y^1-y^2+\int_s^t\Big\{\xi^1(\sigma)-\xi^2(\sigma)-(\sigma N+W^1(\sigma)\Gamma)(x^1(\sigma)-x^2(\sigma))\\ &-(W^1(\sigma)-W^2(\sigma))\Gamma x^2(\sigma)\Big\}d\sigma.\end{aligned}\tag{6.26}$$

This implies

$$\begin{aligned}&|x^1(t,s,y^1,\eta^1)-y^1-(t-s)\eta^1-(x^2(t,s,y^2,\eta^2)-y^2-(t-s)\eta^2)|\\ \le\quad &\int_s^t\Big|\xi^1(\sigma)-\eta^1-(\xi^2(\sigma)-\eta^2)\Big|d\sigma+C_{R,T_0}|t-s||W_1-W_2|_{C([0,T_0])}\\ &+\int_s^t\Big|(\sigma N+W^1(\sigma)\Gamma)(x^1(\sigma)-x^2(\sigma))\Big|d\sigma.\end{aligned}$$

Using again equation (6.26), together with the fact that

$$\begin{aligned}
& \xi^1(\sigma,s,y^1,\eta^1)-\eta^1-(\xi^2(\sigma,s,y^2,\eta^2)-\eta^2) \\
= & \int_s^\sigma (\tau N+W^1(\tau)\Gamma)\Big[\xi^1(\tau)-\xi^2(\tau)-(\tau N+W^1(\tau)\Gamma)(x^1(\tau)-x^2(\tau))\Big]d\tau \\
& +\int_s^\sigma (W^1(\tau)-W^2(\tau))\Gamma\Big[\xi^2(\tau)+(2\tau N-(W^1(\tau)+W^2(\tau))\Gamma)x^2(\tau)\Big]d\tau
\end{aligned}$$

and (6.25), we get for sufficiently small $|t-s|$, depending only on R,

$$\begin{aligned}
& |x^1(t,s,y^1,\eta^1)-y^1-(t-s)\eta^1-(x^2(t,s,y^2,\eta^2)-y^2-(t-s)\eta^2)| \\
\leq\ & C_{R,T_0}|t-s||W_1-W_2|_{C([0,T_0])}+C_{R,T_0}|t-s|^2|\eta^1-\eta^2|+C_{R,T_0}|t-s||y^1-y^2|.
\end{aligned}$$

Thus, for $|t-s|<T_R$ sufficiently small,

$$\begin{aligned}
& |\tilde{x}^1(t,s,y^1,\eta^1)-y^1-\eta^1-(\tilde{x}^2(t,s,y^2,\eta^2)-y^2-\eta^2)| \\
\leq\ & C_{R,T_0}|t-s||W_1-W_2|_{C([0,T_0])}+C_{R,T_0}|t-s|(|\eta^1-\eta^2|+|y^1-y^2|).
\end{aligned}$$

Applying the above inequality with $\eta^j=\tilde{\eta}^j(t,s,y^j,x^j)$ for $j=1,2$, and recalling that $\tilde{x}^j(t,s,y^j,\tilde{\eta}^j(t,s,y^j,x^j))=x^j$, we get

$$\begin{aligned}
& |x^1-y^1-\tilde{\eta}^1(t,s,y^1,x^1)-(x^2-y^2-\tilde{\eta}^2(t,s,y^2,x^2))| \\
\leq\ & C_{R,T_0}|t-s||W_1-W_2|_{C([0,T_0])}+C_{R,T_0}|t-s|(|\tilde{\eta}^1-\tilde{\eta}^2|+|y^1-y^2|). \qquad (6.27)
\end{aligned}$$

Setting $y^1=y^2=0, x^1=x^2=e_k$ in (6.27), we obtain for $|t-s|\leq T\leq 1/2C_{R,T_0}$

$$|\tilde{\eta}^1(t,s,0,e_k)-\tilde{\eta}^2(t,s,0,e_k)|\leq C_{R,T_0}|t-s||W_1-W_2|_{C([0,T_0])}. \qquad (6.28)$$

We use this latter inequality in (6.25) and we get, again for $|t-s|\leq T$ sufficiently small,

$$\begin{aligned}
& |\tilde{\xi}^1(t,s,0,\tilde{\eta}^1(t,s,0,e_k))-\tilde{\xi}^2(t,s,0,\tilde{\eta}^2(t,s,0,e_k))| \\
\leq\ & C_{R,T_0}|t-s||W_1-W_2|_{C([0,T_0])}+C_{R,T_0}|\tilde{\eta}_1(t,s,0,e_k)-\tilde{\eta}_2(t,s,0,e_k)| \\
\leq\ & C_{R,T_0}|t-s||W_1-W_2|_{C([0,T_0])}. \qquad (6.29)
\end{aligned}$$

We now set $x^1=x^2=0$ and $y^1=y^2=e_k$ in (6.27), and we obtain similarly, for $|t-s|$ sufficiently small,

$$|\tilde{\eta}^1(t,s,e_k,0)-\tilde{\eta}^2(t,s,e_k,0)|\leq C_{R,T_0}|t-s||W_1-W_2|_{C([0,T_0])}. \qquad (6.30)$$

Therefore in the same way as above,

$$|\tilde{\xi}^1(t,s,e_k,\tilde{\eta}^1(t,s,e_k,0))-\tilde{\xi}^2(t,s,e_k,\tilde{\eta}^2(t,s,e_k,0))|\leq C_{R,T_0}|t-s||W_1-W_2|_{C([0,T_0])}. \qquad (6.31)$$

It follows then from (3.13),(3.14), (6.28), (6.29) and (6.30), and the linearity of $(\tilde{\xi}(t,s,y,\tilde{\eta}(t,s,y,x)),\tilde{\eta}(t,s,y,x))$ with respect to (x,y) (see (6.9)) that

$$\sum_{|\alpha|+|\beta|=2}\left|\partial_x^\alpha\partial_y^\beta(\tilde{S}^{W_1}(t,s,x,y)-\tilde{S}^{W_2}(t,s,x,y))\right|\leq C_{R,T_0}|t-s||W_1-W_2|_{C([0,T_0])}.$$

Hence,

$$\sum_{|\alpha|+|\beta|=2}\left|\partial_x^\alpha\partial_y^\beta(S^{W_1}(t,s,x,y)-S^{W_2}(t,s,x,y))\right|\leq C_{R,T_0}|W_1-W_2|_{C([0,T_0])}. \qquad (6.32)$$

For the first order derivatives of $S^{W_1}-S^{W_2}$, we have for example, again with the relations (3.13) and (6.9),

$$\begin{aligned}\Big|\partial_{x_l}(S^{W_1}-S^{W_2})(t,s,x,y)\Big| &= \frac{1}{t-s}\Big|\tilde{\xi}_l^1(t,s,y,\tilde{\eta}^1(t,s,y,x))-\tilde{\xi}_l^2(t,s,y,\tilde{\eta}^2(t,s,y,x))\Big|\\ &\le \frac{1}{t-s}\Big|\sum_k y_k[\tilde{\xi}_l^1(t,s,e_k,\tilde{\eta}^1(t,s,e_k,0))-\tilde{\xi}_l^2(t,s,e_k,\tilde{\eta}^2(t,s,e_k,0))]\Big|\\ &\quad+\frac{1}{t-s}\Big|\sum_k x_k[\tilde{\xi}_l^1(t,s,0,\tilde{\eta}^1(t,s,0,e_k))-\tilde{\xi}_l^2(t,s,0,\tilde{\eta}^2(t,s,0,e_k))]\Big|\\ &\le C_{R,T_0}(1+|x|+|y|)|W_1-W_2|_{C([0,T_0])}\end{aligned}$$

where we have used (6.29) and (6.31) in the last inequality. Hence,

$$\sum_{|\alpha|+|\beta|=1}\Big|\partial_x^\alpha\partial_y^\beta(S^{W_1}(t,s,x,y)-S^{W_2}(t,s,x,y))\Big|\le C_{R,T_0}(1+|x|+|y|)|W_1-W_2|_{C([0,T_0])}. \tag{6.33}$$

In order to estimate $S^{W_1}-S^{W_2}$, we write

$$S^{W_j}(t,s,x,y)=S^{W_j}(t,s,0,y)+\int_0^1(\partial_x S^{W_j})(t,s,\theta x,y)\cdot xd\theta,\quad j=1,2,$$

and then we use (6.33) for

$$\Big|\int_0^1\Big((\partial_x S^{W_1})(t,s,\theta x,y)-(\partial_x S^{W_2})(t,s,\theta x,y)\Big)\cdot xd\theta\Big|,$$

which is majorized by

$$C_{R,T_0}|W_1-W_2|_{C([0,T_0])}\int_0^1(1+|\theta x|+|y|)|x|d\theta\le C_{R,T_0}|W_1-W_2|_{C([0,T_0])}(1+|x|^2+|y|^2).$$

Again we develop $S^{W_j}(t,s,0,y)$ around $y=0$, then $S^{W_j}(t,s,0,y)=\int_0^1(\partial_y S^{W_j})(t,s,0,\theta y)\cdot yd\theta$, since $S^{W_j}(t,s,0,0)=0$. We estimate this term as above, and we obtain

$$\Big|S^{W_1}(t,s,x,y)-S^{W_2}(t,s,x,y)\Big|\le C_{R,T_0}|W_1-W_2|_{C([0,T_0])}(1+|x|^2+|y|^2). \tag{6.34}$$

which completes the proof of Lemma 6.1. □

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